GENERALIZED KATO CLASSES AND EXCEPTIONAL ZERO CONJECTURES

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Abstract. The primary objective of this paper is the study of different instances of the *elliptic Stark conjectures* of Darmon, Lauder and Rotger, in a situation where the elliptic curve attached to the modular form f has split multiplicative reduction at p and the arithmetic phenomena are specially rich. For that purpose, we resort to the principle of improved p-adic L-functions and study their $\mathcal{L}$-invariants. We further interpret these results in terms of *derived* cohomology classes coming from the setting of diagonal cycles, showing that the same $\mathcal{L}$-invariant which arises in the theory of p-adic L-functions also governs the arithmetic of Euler systems. Thus, we can reduce, in the split multiplicative situation, the conjecture of Darmon, Lauder and Rotger to a more familiar statement about higher order derivatives of a triple product p-adic L-function at a point lying *inside* the region of classical interpolation, in the realm of the more well-known *exceptional zero conjectures*.

Contents

1. Introduction

The elliptic Stark conjecture was first formulated by Darmon, Lauder and Rotger in [DLR1] as a "more constructive alternative to the Birch and Swinnerton-Dyer conjecture, since it often allows the efficient analytic computation of p-adic logarithms of global points". The conjecture relates a p-adic iterated integral attached to a triple (f, g, h) of cuspidal modular forms with a regulator given in terms of points in an elliptic curve, in a rank 2 situation. Until the moment,

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not too much work towards the proof of the conjecture has been done: most of the results are restricted to situations where there exists a factorization of p-adic L-functions, which allows to interpret the conjecture in terms of the more familiar objects of Bertolini–Darmon–Prasanna [BDP].

However, recent works of Bertolini–Seveso–Veneruci [BSV1], [BSV2] and Darmon–Rotger [DR4], [DR3] suggest an alternative conjecture also in terms of triple product p-adic L-functions: while the first formulation of [DLR1] is concerned with the p-adic value at a point lying *outside* the region of classical interpolation, the *new* version we discuss is about higher order derivatives at a point which belongs to the classical interpolation region. This setting is germane to that explored firstly by Greenberg–Stevens [GS] and then by Bertolini–Darmon [BD2] or Venerucci [Ven]. We propose an alternative conjecture in the split multiplicative setting, and one of the main results of this note is the discussion of the equivalence between both formulations, using for that purpose the setting of generalized cohomology classes. This relies, however, on an apparently deep fact about periods of weight one modular forms, stated in [DR2] as Conjecture 2.1. We believe that this *translation* of the conjecture to a more well understood setting provides new evidence for a better understanding of the problem.

The genesis of this project comes from a parallel story where a new conjecture, formulated in [DLR2], arises; this gives a formula for the p-adic iterated integral when the modular form f is no longer cuspidal, but an Eisenstein series. In [RiRo1], the authors envisaged a method of proof for this conjecture when the two modular forms (g, h) are self-dual: this was based on Hida's improved factorization theorem for the Hida–Rankin p-adic L-function and allowed us to study the conjecture in terms of a question concerning Galois deformations.

The discussion of our results in this paper also leads us to the study of an exceptional vanishing of the generalized cohomology classes of [DR2] and [CaHs], proposing a putative refinement in terms of some *derived* generalized cohomology classes.

Setting and notations. Fix once for all a prime number $p \geq 3$ and three positive integers N_f, N_g, N_h. Let $N = \mathrm{lcm}(N_f, N_g, N_h)$ and assume that $p \nmid N$. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let

$$f \in S_2(pN_f), \quad g \in M_1(N_g, \chi), \quad h \in M_1(N_h, \bar{\chi})$$

be a triple of newforms of weights $(2, 1, 1)$, levels (pN_f, N_g, N_h) and nebentype characters $(1, \chi, \bar{\chi})$, where $\bar{\chi}$ stands for the character obtained by composing χ with complex conjugation. Further, we denote by V_g and by V_h the Artin representations attached to g and h, respectively, and write $V_{gh} := V_g \otimes V_h$. Let H be the number field cut out by this representation, and L for the field over which it is defined. To simplify the exposition, we assume that f has rational Fourier coefficients and that is attached via modularity to an elliptic curve E with split multiplicative reduction at p. Under the assumption that $(pN_f, N_gN_h) = 1$, the global sign of the functional equation of $L(E, V_{gh}, s)$ is $+1$. We keep this assumption from now on.

Label and order the roots of the p-th Hecke polynomial of g as

$$X^2 - a_p(g)X + \chi(p) = (X - \alpha_g)(X - \beta_g)$$

and do the same for those of h. Let $g_\alpha(q) = g(q) - \beta_g(q^p)$ denote the p-stabilization of g with U_p-eigenvalue α_g; it is defined by the q-expansion $g_\alpha(q) = g(q) - \beta_g g(q^p)$. We want to deal with a situation of exceptional zeros, that is, where one or several of the Euler factors involved in the interpolation formula of the p-adic L-function vanish (alternatively, and as we will see later on, this can be understood in terms of the eigenvalues for the Frobenius action). This naturally splits into two different settings, namely

(a) the case where $\alpha_g\alpha_h = 1$ (and therefore $\beta_g\beta_h = 1$); and
(b) the case where $\alpha_g\beta_h = 1$ (and therefore $\beta_g\alpha_h = 1$).

In both cases, if we denote the roots of the p-th Hecke polynomial of g by $\{\alpha_g, \beta_g\}$, those of h are $\{1/\alpha_g, 1/\beta_g\}$. As a piece of notation, we write $h_{1/\alpha}$ and $h_{1/\beta}$ for the p-stabilizations of h with eigenvalues $1/\alpha_g$ and $1/\beta_g$, respectively. Along this work, we refer to these settings as *Case (a)* and *Case (b)*. In the framework of Beilinson–Flach elements and Hida–Rankin p-adic L-functions, the second case has been studied in [RiRo1], and the former has been worked out in [RiRo2, Section 5].

To prove our main results, we also need a classicality property for g. Hence, we assume throughout that

(H1) the reduction of both V_g and V_h modulo p is irreducible (this requires the choice of integral lattices T_g and T_h, but the fact of being irreducible or not is independent of this choice);
(H2) g and h are p-distinguished, i.e, $\alpha_g \neq \beta_g, \alpha_h \neq \beta_h \pmod p$; and
(H3) V_g is not induced from a character of a real quadratic field in which p splits.

Enlarge L if necessary so that it contains all Fourier coefficients of g_α. As shown in [DLR1], the above hypotheses ensure that any generalized overconvergent modular form with the same generalized eigenvalues as g_α is classical, and hence simply a multiple of g_α.

In order to describe our results more precisely, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwasawa algebra and denote by $\mathcal{W} = \mathrm{Spf}(\Lambda)$ the weight space. Hida's theory associates the following data to f:

- a finite flat extension $\Lambda_{\mathbf{f}}$ of Λ, giving rise to a covering $\mathrm{w} : \mathcal{W}_{\mathbf{f}} = \mathrm{Spf}(\Lambda_{\mathbf{f}}) \longrightarrow \mathcal{W}$;
- a family of overconvergent p-adic ordinary modular forms $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ specializing to f at some point $x_0 \in \mathcal{W}_{\mathbf{f}}$ of weight $\mathrm{w}(x_0) = 2$.
- a representation of the absolute Galois group $G_{\mathbb{Q}}$, $\varrho_{\mathbf{f}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(\mathbb{V}_{\mathbf{f}}) \simeq \mathrm{GL}_2(\Lambda_{\mathbf{f}})$ characterized by the property that all its classical specializations coincide with the Galois representation associated by Deligne to the corresponding specialization of the Hida family.

The same occurs with g_α and h_α thanks to the work of Bellaiche and Dimitrov [BeDi] on the geometry of the eigencurve for points of weight one; we denote by $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ the corresponding extensions of Λ over which the Hida families $\mathbf{g}$ and $\mathbf{h}$ are defined, and by $y_0 \in \mathcal{W}_{\mathbf{g}}$, $z_0 \in \mathcal{W}_{\mathbf{h}}$ the weight one points for which the corresponding specializations agree with g_α and h_α, respectively.

For each of the settings (a) and (b) presented above, we discuss three different objects which are expected to encode arithmetic information regarding the convolution of the three Galois representations attached to the modular forms f, g and h. We denote by (x, y, z) a triple of points in $\mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$, whose weights are referred as (k, ℓ, m).

(i) The cohomology classes $\kappa(f, g_\alpha, h_\alpha)$ studied for instance in [DR2] and [CaHs], arising as the specialization at weights $(2,1,1)$ of the three-variable family $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ constructed as the image under a p-adic Abel-Jacobi map of certain diagonal cycles. In general, one may construct four different classes

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta),$$

one for each p-stabilization of g and h. Further, when some of these classes vanish, we are lead to consider their derivatives.

(ii) The special value $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at weights $(2,1,1)$ and its derivatives. Here, $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ stands for the three-variable p-adic L-function attached to three Hida families, characterized by an interpolation property regarding the classical values of the triple product L-function at the region where $k \geq \ell + m$. When this function vanishes at the point $(2,1,1)$, the derivatives along different directions of the weight space may encode *interesting* arithmetic information.

(iii) The special value $\mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})$ at weights $(2,1,1)$, denoted $\mathscr{L}_p{}^{g_\alpha}$. This p-adic L-function is defined in an analogue way to the previous one, but now the region of interpolation is characterized by the inequality $\ell \geq k+m$ so the point $(2,1,1)$ is outside the region of classical interpolation. Similarly, we may also take $\mathscr{L}_p{}^{h_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})$, whose region of interpolation concerns those points for which $m \geq k+\ell$. Observe that the first value depends on the choice of p-stabilizations for the weight one form g_α.

(i) Cohomology classes coming from the theory of diagonal cycles. We begin by recalling the results concerning cohomology classes. Results of this kind had already been explored in [BSV2] and [DR3] when $\alpha_g\alpha_h = 1$. In that case, the cohomology class is not expected to vanish, but the numerator of the (Perrin-Riou) regulator in the reciprocity law for $\mathscr{L}_p{}^f$ does, which is coherent with the fact that the p-adic L-function $\mathscr{L}_p{}^f(f,g,h)$ is zero (this can be seen, of course, as an exceptional zero coming from the vanishing of an Euler factor).

Here we are mostly interested in the case where the denominator of the Perrin-Riou regulator in the reciprocity law for $\mathscr{L}_p{}^{g_\alpha}$ vanishes due to another exceptional zero phenomenon. This occurs when $\alpha_g\beta_h = 1$ and leads us to recover the ideas of [Cas], [RiRo1] and [Ri1], where this same phenomenon was studied for Heegner points, Beilinson–Flach elements and elliptic units, respectively. In those cases, the reciprocity laws linking Euler systems and p-adic L-functions were updated to *derived reciprocity laws.* A different approach is taken also in [BSV1, Section 9], where the authors introduce certain *improved* cohomology classes, which in this case we may compare in an explicit way with appropriate *derived* elements.

Define the three-variable Iwasawa algebra $\Lambda_{\mathbf{fgh}} := \Lambda_{\mathbf{f}}\hat{\otimes}_{\mathbb{Z}_p}\Lambda_{\mathbf{g}}\hat{\otimes}_{\mathbb{Z}_p}\Lambda_{\mathbf{h}}$ and the $\Lambda_{\mathbf{fgh}}[G_{\mathbb{Q}}]$-module

$$\mathbb{V}_{\mathbf{fgh}} := \mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{V}_{\mathbf{h}}.$$

We work with $\mathbb{V}^\dagger_{\mathbf{fgh}}$, a certain twist of it by an appropriate power of the Λ-adic cyclotomic character defined for instance in [DR3, Section 5.1] and that is needed to satisfy the self-dual assumption.

The works [BSV1] and [DR3] attach to $(\mathbf{f},\mathbf{g},\mathbf{h})$ a Λ-adic global cohomology class

$$\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q},\mathbb{V}^\dagger_{\mathbf{fgh}})$$

parameterized by the triple product of the weight space $\mathcal{W}_{\mathbf{fgh}} := \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$.

Consider the specialization of the class at weights (x_0,y_0,z_0),

$$\kappa(f,g_\alpha,h_{1/\beta}) \in H^1(\mathbb{Q},V_{fgh}),$$

where V_{fgh} is the tensor product $V_f \otimes V_g \otimes V_h$ of the Galois representations attached to the modular forms f, g and h. This class can be shown to be trivial and hence we are placed to work with an appropriate *derived* class $\kappa'(f,g_\alpha,h_{1/\beta})$.

As it occurred in the setting of Beilinson–Flach classes, the notion of derivative is rather flexible. Following [RiRo1], we consider here a derivative along an analytic direction, and keeping fixed the weight of h. Rather informally, this may be thought as the line $(\ell+1,\ell,1)$ of the weight space. Note that at least in the self-dual case, where we may argue that the corresponding class vanishes all along the line $(2,\ell,\ell)$, we may consider the derivative along any direction of the weight space.

Let $\alpha_{\mathbf{f}}$ (resp. $\alpha_{\mathbf{g}}$, $\alpha_{\mathbf{h}}$) stand for the Iwasawa function corresponding to the root of the p-th Hecke polynomial of $\mathbf{f}$ (resp. $\mathbf{g}$, $\mathbf{h}$) with smallest p-adic valuation. As an additional piece of notation, let

$$\mathcal{L} := \frac{\alpha_g'}{\alpha_g} - \frac{\alpha_f'}{\alpha_f}, \tag{1}$$

where α'_f (resp. α'_g, α'_h) stands for the derivative of the Frobenius eigenvalues at x_0 (resp. y_0, z_0) when seen as an Iwasawa function along the Hida family $\Lambda_{\mathbf{f}}$ (resp. $\Lambda_{\mathbf{g}}$, $\Lambda_{\mathbf{h}}$). Observe that we can give explicit formulas for $\mathcal{L}$, involving both some units and p-units in the field cut out by the representation V_{gh} and the Tate uniformizer of the elliptic curve E. Hence, the $\mathcal{L}$-invariant governing the arithmetic of the triple (f,g,h) is related both with the $\mathcal{L}$-invariant of the elliptic curve (the logarithm of the Tate uniformizer) and also with the regulator attached to the adjoint representation $\mathrm{ad}^0(V_g)$, expressed in [RiRo1] as a combination of logarithms of units and p-units. Compare for instance this result with the main theorem of [Cas], where he interprets the $\mathcal{L}$-invariant attached to a modular form f and an anticyclotomic character as the sum of the two $\mathcal{L}$-invariants. Our first main result is the following (see Theorem 3.10 for the precise formulation), relating an appropriate logarithm of the *derived* class with the special value $\mathscr{L}_p{}^{g_\alpha}$.

Theorem 1.1. *The derived local cohomology class satisfies*

$$\langle \log_{\mathrm{BK}}(\kappa'_p(f,g_\alpha,h_{1/\beta})^g), \omega_f \otimes \eta_{g_\alpha} \otimes \omega_{h_{1/\beta}} \rangle = \mathcal{L} \cdot \mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})(x_0,y_0,z_0) \pmod{L^\times},$$

where the superindex g stands for an appropriate projection of κ'_p that we later introduce, and $\log_{\mathrm{BK}}$ refers to the Bloch–Kato logarithm, followed by the pairing $\langle -,- \rangle$ with certain canonical differentials.

Remark. *In* [BSV1] *the authors take a different approach to this exceptional zero phenomenon, and construct an improved cohomology class $\kappa_g^*(f,g_\alpha,h_{1/\beta})$. As we will later show, there is a connection between both constructions and one may prove (under mild conditions!) that the following equality holds in $H^1(\mathbb{Q},(V_f \otimes V_g \otimes V_h)_{|\mathcal{S}})$, where $\mathcal{S}$ stands for the subvariety of the weight space along which the derived and the improved class are defined, corresponding to the set of weights $k+m=\ell+2$:*

$$\kappa'(f,g_\alpha,h_{1/\beta}) = \mathcal{L} \cdot \kappa_g^*(f,g_\alpha,h_{1/\beta}). \tag{2}$$

(ii) The special value $\mathscr{L}_p{}^f$ and derivatives of the triple product p-adic L-function. In subsequent parts of the article we use the previous cohomology classes to study different instances of the elliptic Stark conjecture. Section 4 is devoted to analyze higher order derivatives of $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$ at (x_0,y_0,z_0). The presence of an Euler factor which vanishes at weights $(2,1,1)$ automatically forces the vanishing of that value. Therefore, it is natural to formulate several conjectures for the value of the derivatives of $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$.

When $\alpha_g\alpha_h=1$ and $L(f\otimes g\otimes h,1)\neq 0$, the results of Bertolini–Seveso–Venerucci relying on the existence of an improved p-adic L-function allow us to state the following result. Although this can be seen as a straightforward corollary of the results developed in loc. cit., we want to point out that the $\mathcal{L}$-invariants attached to both g and h have a strong connection with the arithmetic of number fields. This reveals that in the rank 0 situation the quantity $\mathscr{L}_p{}^f$ is also a putative refinement of the more well-known $\mathcal{L}$-invariants of Greenberg–Stevens, where not only the Tate period q_E appears. This result follows from [BSV1, Proposition 9.3].

Proposition (Bertolini–Seveso–Venerucci)**.** *Let I denote the ideal of functions in $\Lambda_{\mathbf{fgh}}$ which vanish at (x_0,y_0,z_0). Assume that $L(f\otimes g\otimes h,1)\neq 0$, and let $\mathcal{L}_\xi := \alpha'_\xi/\alpha_\xi$, for $\xi \in \{f,g_\alpha,h_\alpha\}$. Then, up to a constant in $L^\times$,*

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}) = (\mathcal{L}_{g_\alpha}-\mathcal{L}_f)(\ell-1)+(\mathcal{L}_{h_\alpha}-\mathcal{L}_f)(m-1) \pmod{I^2}.$$

Moreover, the quantities $\mathcal{L}_\chi$ are explicitly computable in terms of the arithmetic of number fields and elliptic curves.

Observe for example that the derivative along the y-direction agrees with the $\mathcal{L}$-invariant that also arises as the derivative of the diagonal class discussed before.

However, the most interesting case appears when $L(f \otimes g \otimes h, 1) = 0$. Let us put ourselves in the setting of [DLR1] and assume that $(E(H) \otimes V_{gh})^{\mathrm{Gal}(H/\mathbb{Q})}$ is two-dimensional. This group is equipped with an inclusion in the p-adic Selmer group corresponding to the group of extensions of $\mathbb{Q}_p$ by V_{fgh} in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$ that are crystalline at p. This group is denoted by $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$, and we also assume that is two-dimensional (the latter would follow from the Birch and Swinnerton-Dyer conjecture for the pair (E, V_{gh}) and the finiteness of the corresponding Tate–Shafarevich group).

Let $\{P, Q\}$ denote generators of $(E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}$, and fix a basis $\{e_{\alpha\alpha}, e_{\alpha\beta}, e_{\beta\alpha}, e_{\beta\beta}\}$ of V_{gh} as a $G_{\mathbb{Q}_p}$-module with the Frobenius action. This allows us to write

$$P = P_{\beta\beta} \otimes e_{\beta\beta} + P_{\beta\alpha} \otimes e_{\beta\alpha} + P_{\alpha\beta} \otimes e_{\alpha\beta} + P_{\alpha\alpha} \otimes e_{\alpha\alpha},$$

and similarly for Q. Here, the arithmetic Frobenius Fr_p acts on $P_{\beta\beta}$ with eigenvalue $\beta_g\beta_h$ and analogously for the remaining components. In this scenario, we can conjecture the following result, that we extensively discuss in Section 5.

Conjecture 1.2. *Assume that the L-dimension of $(E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}$ is two. Then, under the running assumptions, the p-adic L-function $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ satisfies*

$$\frac{\partial^2 \mathscr{L}_p{}^f(\mathbf{f}_x, g_\alpha, h_{1/\alpha})}{\partial x^2}\Big|_{x=x_0} = \log_p(P_{\beta\beta}) \cdot \log_p(Q_{\alpha\alpha}) - \log_p(Q_{\beta\beta}) \cdot \log_p(P_{\alpha\alpha}) \pmod{L^\times}.$$

If the L-dimension of $(E(H) \otimes V_{gh})^{G_{\mathbb{Q}}}$ is greater than two, then the left hand side vanishes.

There are other interesting lines along weight space to take derivatives. For example, in [CaHs] the study is concerned with the line $(2, \ell, \ell)$, where the derivatives are connected with appropriate *derived* heights of the points P and Q.

The work of Bertolini–Seveso–Venerucci and Darmon–Rotger establishes the conjecture for the case where g and h are theta series of a quadratic field where p is inert, which leads to a decomposition $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$. In the imaginary case, we can extend their computations to the split case, observing that here one has a trivial equality of the form $0 = 0$.

Therefore, we may establish that Conjecture 1.2 holds in some dihedral cases. The first part of this Proposition follows from [BSV2, Theorem A], and the second is established as part of Proposition 4.7.

Proposition 1.3. *Conjecture 1.2 holds in the following cases:*

(a) *CM or RM series with p inert in K and at least one of ψ_1 or ψ_2 being a genus characters;*
(b) *CM series with p split in K, $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, with each component of rank one.*

We must say that in all these cases the proof is based on a factorization formula, so we expect that new ideas would be required for the proof in the general case.

(iii) The special value $\mathscr{L}_p{}^{g_\alpha}$. In the last section, we discuss a way to connect the previous conjecture with the elliptic Stark conjecture of [DLR1] when $\alpha_g\alpha_h = 1$. Recall that the conjecture predicts that

$$\mathscr{L}_p{}^{g_\alpha}(f, g_\alpha, h_\alpha) = \frac{\log_p(P_{\beta\alpha})\log_p(Q_{\beta\beta}) - \log_p(P_{\beta\beta})\log_p(Q_{\beta\alpha})}{\log_p(u_{g_\alpha})} \pmod{L^\times}, \tag{3}$$

with u_{g_α} being a Gross–Stark unit whose characterization we later recall. In particular, it is expected that this unit could be expressed as a ratio of periods attached to weight one forms. These two periods, denoted by Ω_{g_α} and Ξ_{g_α}, will play a prominent role in the last part of the work. More precisely, in [DR2, eq. (9)], the authors introduce a p-adic period, $\mathcal{L}_{g_\alpha} = \Omega_{g_\alpha}/\Xi_{g_\alpha}$ and conjecture (see Conjecture 2.1 of loc. cit.)

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}). \tag{4}$$

In Section 5 we consider the following three conjectures:

(i) the elliptic Stark conjecture for ${\mathscr{L}_p}^{g_\alpha}$;
(ii) the conjecture for the second derivative along the f-direction for ${\mathscr{L}_p}^{f}$, i.e., Conjecture 1.2;
(iii) [DR2, Conjecture 2.1] about periods of weight one modular forms. Proposition 5.1 can be seen as an extra piece of theoretical evidence towards this conjecture, showing that

$$\frac{\mathcal{L}_{g_\alpha}}{\mathcal{L}_{g_\beta}} = \frac{\log_p(u_{g_\alpha})}{\log_p(u_{g_\beta})}.$$

Under certain non-vanishing hypothesis, we prove that if two of the previous conjectures are true, the third one automatically holds. In particular, we establish the following in Corollary 5.5.

Theorem 1.4. *Let g and h be theta series of a quadratic field (either real or imaginary) where p is inert. Write $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, and assume that either ψ_1 or ψ_2 is a genus character. Then, under the given assumptions, the equality* (4) *is equivalent to the elliptic Stark conjecture of Darmon, Lauder and Rotger* (3)*.*

All the previous results are based on the interaction of the different arithmetic actors when $\alpha_g \alpha_h = 1$. The case where $\alpha_g \beta_h = 1$ is more subtle, since here the cohomology class $\kappa(f, g_\alpha, h_{1/\beta})$ vanishes and we cannot extract the same arithmetic information. In any case, we expect that a similar result must hold in this setting. The reason is that the value of ${\mathscr{L}_p}^{g_\alpha}$ does not depend on the choice of a p-stabilization for h, and hence we can also give a conjectural expression for the derived cohomology class in terms of points, in complete analogy with [RiRo1, Theorem B].

Conjecture 1.5. *The following equality holds in $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$:*

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \frac{\mathcal{L}}{\Xi_{g_\alpha} \cdot \Omega_{h_{1/\beta}}} \cdot \frac{\log_p(P_{\beta\beta}) \cdot Q - \log_p(Q_{\beta\beta}) \cdot P}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{5}$$

Proceeding as in [DR2] and [RiRo2] we may also obtain expressions (at least conjecturally) for the three remaining cohomology classes, $\kappa'(f, g_\beta, h_{1/\alpha})$, $\kappa(f, g_\alpha, h_{1/\alpha})$ and $\kappa(f, g_\beta, h_{1/\beta})$.

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2. Preliminaries

This section aims to give an overview of the setting we present, concerned with triple product p-adic L-functions, and also recalls some known results in other related scenarios coming from the theory of elliptic curves and weight one modular forms.

2.1. Hsieh's triple product p-adic L-function.

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For a number field K, let $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ denote its absolute Galois group. Fix also an odd prime p and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

The formal spectrum $\mathcal{W} = \mathrm{Spf}(\Lambda)$ of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is called the weight space attached to Λ. The weight space is equipped with a distinguished class of *arithmetic* *points* $\nu_{s,\varepsilon}$ indexed by integers $s \in \mathbb{Z}$ and Dirichlet characters $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ of p-power conductor. The point $\nu_{s,\varepsilon} \in \mathcal{W}$ is defined by the rule

$$\nu_{s,\varepsilon}(n) = \varepsilon(n)n^s.$$

Let $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ be a triple of p-adic Hida families of tame levels $N_\mathbf{f}$, $N_\mathbf{g}$, $N_\mathbf{h}$ and tame characters $\chi_\mathbf{f}$, $\chi_\mathbf{g}$, $\chi_\mathbf{h}$. Assume that $\chi_\mathbf{f}\chi_\mathbf{g}\chi_\mathbf{h} = 1$ (this is referred to as the self-duality assumption). Set $N = \mathrm{lcm}(N_\mathbf{f}, N_\mathbf{g}, N_\mathbf{h})$, and suppose that $p \nmid N$.

Let $\Lambda_\mathbf{f}$, $\Lambda_\mathbf{g}$ and $\Lambda_\mathbf{h}$ be the finite extensions of Λ generated by the coefficients of the Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, respectively. The weight space attached to $\Lambda_\mathbf{f}$ is $\mathcal{W}_\mathbf{f} := \mathrm{Spf}(\Lambda_\mathbf{f})$. Since $\Lambda_\mathbf{f}$ is a finite flat algebra over Λ, there is a natural finite map

$$\pi : \mathcal{W}_\mathbf{f} := \mathrm{Spf}(\mathcal{W}_\mathbf{f}) \xrightarrow{\mathrm{w}} \mathcal{W},$$

and we say that a point $x \in \mathcal{W}_\mathbf{f}$ is arithmetic of weight s and character ε if $\pi(x) = \nu_{s,\varepsilon}$.

A point $x \in \mathcal{W}_\mathbf{f}$ of weight $k \geq 1$ and character ε is said to be crystalline if $\varepsilon = 1$ and there exists an eigenform f_x° of level N such that f_x is the ordinary p-stablization of f_x°. We denote by $\mathcal{W}_\mathbf{f}^\circ$ the set of crystalline arithmetic points of $\mathcal{W}_\mathbf{f}$.

Finally, set $\Lambda_\mathbf{fgh} = \Lambda_\mathbf{f}\hat{\otimes}\Lambda_\mathbf{g}\hat{\otimes}\Lambda_\mathbf{h}$ and let $\mathcal{W}_\mathbf{fgh}^\circ := \mathcal{W}_\mathbf{f}^\circ \times \mathcal{W}_\mathbf{g}^\circ \times \mathcal{W}_\mathbf{h}^\circ \subset \mathcal{W}_\mathbf{fgh} = \mathrm{Spf}(\Lambda_\mathbf{fgh})$ be the set of triples of crystalline classical points, at which the three Hida families specialize to modular forms with trivial nebentype at p. This set admits the natural partition

$$\mathcal{W}_\mathbf{fgh}^\circ = \mathcal{W}_\mathbf{fgh}^f \sqcup \mathcal{W}_\mathbf{fgh}^g \sqcup \mathcal{W}_\mathbf{fgh}^h \sqcup \mathcal{W}_\mathbf{fgh}^{\mathrm{bal}},$$

where

- $\mathcal{W}_\mathbf{fgh}^f$ denotes the set of points $(x, y, z) \in \mathcal{W}_\mathbf{fgh}^\circ$ of weights (k, ℓ, m) such that $k \geq \ell + m$.
- $\mathcal{W}_\mathbf{fgh}^g$ and $\mathcal{W}_\mathbf{fgh}^h$ are defined similarly, replacing the role of $\mathbf{f}$ by $\mathbf{g}$ (resp. $\mathbf{h}$).
- $\mathcal{W}_\mathbf{fgh}^{\mathrm{bal}}$ is the set of balanced triples, consisting of points (x, y, z) of weights (k, ℓ, m) such that each of the weights is strictly smaller than the sum of the other two.

Recall from [DR3, Section 1.4] the notion of test vector. As proved in Section 3.5 of loc. cit. following [Hs], there is a canonical choice of test vectors for which there exists a *square-root* p-adic L-function

$$\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) : \mathcal{W}_\mathbf{fgh} \to \mathbb{C}_p,$$

characterized by an interpolation property relating its values at classical points $(x, y, z) \in \mathcal{W}_\mathbf{fgh}^f$ to the square root of the central critical value of Garrett's triple-product complex L-function $L(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z, s)$ associated to the triple of classical eigenforms $(\mathbf{f}_x, \mathbf{g}_y, \mathbf{h}_z)$. For the following proposition, let $\alpha_{\mathbf{f}_x}$ and $\beta_{\mathbf{f}_x}$ be the roots of the p-th Hecke polynomial of $\mathbf{f}_x$, ordered in such a way that $\mathrm{ord}_p(\alpha_{\mathbf{f}_x}) \leq \mathrm{ord}_p(\beta_{\mathbf{f}_x})$. The following result is [DR3, Proposition 5.1].

Proposition 2.1. *Fix test vectors* $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ *as in* [Hs, Section 3]. *Then* $\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})$ *lies in* $\Lambda_\mathbf{fgh}$ *and for every* $(x, y, z) \in \mathcal{W}_\mathbf{fgh}^f$ *of weights* (k, ℓ, m) *we have*

$$\mathscr{L}_p{}^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})^2(x, y, z) = \frac{\mathfrak{a}(k, \ell, m)}{\langle \mathbf{f}_x^\circ, \mathbf{f}_x^\circ \rangle^2} \cdot \mathfrak{e}^2(x, y, z) \times L(\mathbf{f}_x^\circ, \mathbf{g}_y^\circ, \mathbf{h}_z^\circ, c),$$

where

(1) $c = \frac{k+\ell+m-2}{2}$.
(2) $\mathfrak{a}(k, \ell, m) = (2\pi i)^{-2k} \cdot \left(\frac{k+\ell+m-4}{2}\right)! \cdot \left(\frac{k+\ell-m-2}{2}\right)! \cdot \left(\frac{k-\ell+m-2}{2}\right)! \cdot \left(\frac{k-\ell-m}{2}\right)!,$

(3) $\mathfrak{e}(x,y,z) = \mathcal{E}(x,y,z)/\mathcal{E}_0(x)\mathcal{E}_1(x)$ *with*

$$\begin{aligned}
\mathcal{E}_0(x) &:= 1-\chi_f^{-1}(p)\beta_{\mathbf{f}_x}^2 p^{1-k},\\
\mathcal{E}_1(x) &:= 1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-2} p^{k-2},\\
\mathcal{E}(x,y,z) &:= \Big(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\Big)\times\Big(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\Big)\\
&\quad\times\Big(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\Big)\times\Big(1-\chi_f(p)\alpha_{\mathbf{f}_x}^{-1}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\Big).
\end{aligned}$$

For simplicity, we fix once and for all these test vectors and remove their dependence in the notation. Moreover, and to be more consistent with other papers in the topic, we may safely shrink the weight space and restrict to a fixed congruence class of weights modulo $2(p-1)$.

There is an ostensible parallelism between this p-adic L-function and the so-called Hida–Rankin p-adic L-function attached to a pair of Hida families $(\mathbf{g},\mathbf{h})$, but where the cyclotomic variable s is allowed to move freely. It may be instructive to keep in mind this analogy for the subsequent results.

Some of the easiest cases to understand these triple product p-adic L-functions arise when the representation attached to V_{gh} is irreducible. In particular, assume that g is a weight one theta series attached to a quadratic field K (either real of imaginary) where p remains inert. Then, $V_{gh} = V_{\psi_1}\oplus V_{\psi_2}$, and under the assumption that at least one between ψ_1 or ψ_2 is a genus (quadratic) character, the works [BSV2] and [DR3] show that

$$\mathscr{L}_p{}^f(\mathbf{f},g,h)^2 = \mathfrak{f}(k)\cdot L_p(\mathbf{f}/K,\psi_1)\cdot L_p(\mathbf{f}/K,\psi_2), \tag{6}$$

where $\mathfrak{f}(k)$ is a bounded analytic function on $\Lambda_{\mathbf{f}}$ such that $\mathfrak{f}(x_0)\in L^\times$. Here, $\mathscr{L}_p(\mathbf{f}/K,\psi)$ is the two-variable p-adic L-function attached to a Hida family $\mathbf{f}$ and a character ψ of a quadratic field.

As a word of caution, observe that there are *three* different p-adic L-functions, depending on the region of classical interpolation (associated to the dominant weight).

2.2. **Improved p-adic L-functions.** It is a natural phenomenon in the study of p-adic L-functions that some of the Euler factors arising in the interpolation process are analytic along a subvariety of the weight space. When this happens, one is tempted to define *improved* p-adic L-functions, that is, functions over the corresponding subvariety characterized by the same interpolation property, but with these Euler factors removed. This is a quite well-known phenomenon, which dates back to Greenberg–Stevens [GS] and their study of the Mazur–Kitagawa p-adic L-function. This was one of the key ingredients in the proof of our main results in [RiRo1] and we would like to stress the limitations of the method in this triple product setting. The interest of this study is that we also want to discuss later on its applicability from the Euler system side in order to construct *improved cohomology classes.*

For the sake of simplicity, assume that χ_f is trivial. In the setting of triple product p-adic L-functions we have just discussed, one of the Euler factors appearing in the interpolation property of $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$ is

$$1-\frac{\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{\alpha_{\mathbf{f}_x}}p^{\frac{k-\ell-m}{2}},$$

which is an Iwasawa function along the surface $\mathcal{S}_{k=\ell+m}$ defined by

$$\mathcal{S}_{k=\ell+m} = \{(x,y,z)\in\mathcal{W}_{\mathbf{f}}^\circ\times\mathcal{W}_{\mathbf{g}}^\circ\times\mathcal{W}_{\mathbf{h}}^\circ \text{ such that } k=\ell+m\}.$$

The definitions given in [DR1, Def. 4.4] can be adapted to yield an *improved* p-adic L-function $\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})^*$ on $\mathcal{S}_{k=\ell+m}$, by replacing the family $\mathbf{h}\times d^t\mathbf{g}^{[p]}$ with the family $\mathbf{h}\times\mathbf{g}$, whose coefficients vary analytically because $t=0$ on $\mathcal{S}_{k=\ell+m}$.

Proposition 2.2. *There exists an analytic p-adic L-function over the surface $\mathcal{S}_{k=\ell+m}$, denoted by $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$, such that the following equality holds in $\mathcal{S}_{k=\ell+m}$:*

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}) = (1-\alpha_{\mathbf{f}}^{-1}\alpha_{\mathbf{g}}\alpha_{\mathbf{h}})\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*. \tag{7}$$

Proof. This follows from the proof of [BSV1, Proposition 9.3] and the discussion after it. □

We point out that the improved p-adic L-function we have considered, $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$, interpolates classical L-values, in the same way than $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$, but with the vanishing Euler factor removed. Therefore, its value at (x_0,y_0,y_0) is given by an explicit non-zero multiple of the square root of the algebraic part of $L(f,g,h,1)$. In particular, $L(f,g,h,1)\neq 0$ if and only if the improved p-adic L-function does not vanish at (x_0,y_0,y_0).

Observe however that we may also consider other Euler factors. Take for example

$$1-\frac{\bar{\chi}_h(p)\alpha_{\mathbf{h}_z}}{\alpha_{\mathbf{g}_y}\alpha_{\mathbf{f}_x}}p^{\frac{k+\ell-m-2}{2}},$$

which is analytic along $k+\ell=m+2$.

We would expect that one can establish that these factors (each along its respective region) divide the p-adic L-function and yield other improved p-adic L-functions satisfying *mild* analytic and interpolation properties.

2.3. **Exceptional zeros and L-invariants.** The situations we study in this article are mostly concerned with the so-called exceptional zero phenomenon. We now recall several results which appear in the literature around that, mainly in [GS], [Ven] and [RiRo1]. As anticipated before, the point is that the $\mathcal{L}$-invariant governing the arithmetic of $V_f\otimes V_{gh}$ is a combination of the $\mathcal{L}$-invariants attached to f and the adjoint of g (or the adjoint of h, according to the *direction* we choose). For a brief summary of the usual definition and the main properties of the adjoint representation in this scenario, see [DLR1, Section 1.2].

The aim of this section is to give an arithmetic description of the different $\mathcal{L}$-invariants that later appear in the setting of triple products, to have a complete description of our picture.

Let us define, to ease notations,

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) := \frac{\alpha_g'}{\alpha_g}, \qquad \mathcal{L}(E) := \frac{\alpha_f'}{\alpha_f}, \tag{8}$$

where the derivative is taken along the unique Hida family passing through g_α and f, respectively, and then evaluating at the points corresponding to g_α and f.

I. The $\mathcal{L}$-invariant of the adjoint of a modular form. One of the main results of [RiRo1] was the computation of the $\mathcal{L}$-invariant associated to the adjoint of a modular form.

As shown in [DLR2, Lemma 1.1], we have

$$\dim_L(\mathcal{O}_H^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}=1,\quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z}\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}=2.$$

Fix a generator u of $(\mathcal{O}_H^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}$ and also an element v of $(\mathcal{O}_H^\times[1/p]^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}$ such that $\{u,v\}$ is a basis of $(\mathcal{O}_H[1/p]^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}$. The element v may be chosen to have p-adic valuation $\mathrm{ord}_p(v)=1$, and we do so. Viewed as a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g)=L^1\oplus L^{\alpha/\beta}\oplus L^{\beta/\alpha}$, where all the summands are 1-dimensional subspaces characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β and β/α, respectively. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$ and let

$$u_1,\ u_{\alpha/\beta},\ u_{\beta/\alpha},\ v_1,\ v_{\alpha/\beta},\ v_{\beta/\alpha}\in H_p^\times\otimes_\mathbb{Q} L \qquad (\mathrm{mod}\ L^\times)$$

denote the projection of the elements u and v in $(H_p^\times\otimes\mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines. By construction we have $u_1,v_1\in\mathbb{Q}_p^\times$ and

$$\mathrm{Fr}_p(u_{\alpha/\beta})=\frac{\beta}{\alpha}u_{\alpha/\beta},\quad \mathrm{Fr}_p(v_{\alpha/\beta})=\frac{\beta}{\alpha}v_{\alpha/\beta},\quad \mathrm{Fr}_p(u_{\beta/\alpha})=\frac{\alpha}{\beta}u_{\beta/\alpha},\quad \mathrm{Fr}_p(v_{\beta/\alpha})=\frac{\alpha}{\beta}v_{\beta/\alpha}.$$

Let

$$\log_p : H_p^\times \otimes L \longrightarrow H_p \otimes L$$

denote the usual p-adic logarithm.

Then, one the main results of [RiRo1] was the computation of $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$, which can be expressed as

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = -\frac{\log_p(v_1)\log_p(u_{\alpha/\beta}) - \log_p(u_1)\log_p(v_{\alpha/\beta})}{2\mathrm{ord}_p(v_1)\cdot \log_p(u_{\alpha/\beta})}. \tag{9}$$

II. The $\mathcal{L}$-invariant of an elliptic curve (rank 0). In [GS], the authors prove a conjecture of Mazur, Tate and Teitelbaum [MTT] expressing the quantity $L_p(E,1)$ in terms of the derivative of $L(E,1)$ when the rank is zero. As a consequence of this, they show that an elliptic curve with split multiplicative reduction at p satisfies

$$\mathcal{L}(E) = -\frac{\log_p(q_E)}{2\mathrm{ord}_p(q_E)}, \tag{10}$$

where q_E is Tate's uniformizer for the elliptic curve E. We write $L_p(\mathbf{f})(x,s)$ for the usual two-variable Mazur–Kitagawa p-adic L-function, and x_0 for the weight two point satisfying $\mathbf{f}_{x_0} = f$, with f the modular form attached to E by modularity.

As recalled for instance in the discussion of [BD2, Remark 1.13], there exists an improved p-adic L-function along $s=1$, that we denote here as $L_p^*(\mathbf{f})(x)$ and which is characterized by

$$L_p(\mathbf{f})(x,1) = (1 - a_p(\mathbf{f}_x)^{-1})\cdot L_p^*(\mathbf{f})(x).$$

Observe that in a rank 0 situation $L_p^*(\mathbf{f}_{x_0})$ is a non-zero algebraic number which agrees (up to constant) with the algebraic part of the classical L-value.

III. The $\mathcal{L}$-invariant of an elliptic curve (rank 1). In a rank 1 situation, Venerucci relates the second derivatives of the Mazur–Kitagawa p-adic L-function with certain heights of Heegner points. Observe that in this setting, $L_p(\mathbf{f})(x_0,1) = 0$ and the same happens for its first derivatives. To determine the second order derivatives, he recasts in [Ven] to the theory of Selmer complexes and Nekovář's Selmer groups, as introduced in [Nek].

Following the conventions used in loc. cit., let $\tilde{H}^1_{\mathrm{f}}$ be Nekovar's extended Selmer group. It is a $\mathbb{Q}_p$-module, equipped with a natural inclusion of the extended Mordell-Weil group of E, that we denote by $E^\dagger(\mathbb{Q})\otimes\mathbb{Q}_p$. In general,

$$\tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) = H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \oplus \mathbb{Q}_p\cdot q_E,$$

where $H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$ is the Bloch–Kato p-adic Selmer group. Using Nekovar and Venerucci's results, there is a canonical $\mathbb{Q}_p$-bilinear form

$$\langle\cdot,\cdot\rangle : \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \otimes_{\mathbb{Q}_p} \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \to I/I^2,$$

where I stands for the augmentation ideal of the cyclotomic Iwasawa algebra, and which may be thought as the ring of functions vanishing at the point $(x,s) = (x_0,1)$, that with a slight abuse of notation we denote by $(2,1)$. This is the so-called *height-weight* pairing, which decomposes as

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_p^{\mathrm{cyc}}\cdot\{s-1\} + \langle\cdot,\cdot\rangle_p^{\mathrm{wt}}\cdot\{k-2\},$$

where $\langle\cdot,\cdot\rangle_p^{\mathrm{cyc}}$ and $\langle\cdot,\cdot\rangle_p^{\mathrm{wt}}$ are canonical $\mathbb{Q}_p$-valued pairings on the extended Selmer group. Finally, the Schneider height is defined by

$$\langle\cdot,\cdot\rangle_p^{\mathrm{Sch}} = \langle x,y\rangle_p^{\mathrm{cyc}} - \frac{\log_p(\mathrm{res}_p(x))\cdot\log_p(\mathrm{res}_p(y))}{\log_p(q_E)},$$

where $\mathrm{res}_p(x)$ is the localization-at-p map. The following result provides expressions for the second derivative of $L_p(\mathbf{f})$ along different directions of the weight space.

Proposition 2.3. *The following formulas hold, where P is a generator of the Mordell–Weil group $E(\mathbb{Q})$.*

(a)
$$\frac{d^2 L_p(\mathbf{f})(k,k/2)}{dk^2}\Big|_{k=2} = \log_E(P)^2 \pmod{L^\times};$$

(b)
$$\frac{d^2 L_p(\mathbf{f})(k,1)}{dk^2}\Big|_{k=2} = \mathcal{L}(E)\cdot \langle P,P\rangle^{\mathrm{cyc}} \pmod{L^\times};$$

(c)
$$\frac{d^2 L_p(\mathbf{f})(x_0,s)}{ds^2}\Big|_{s=1} = \mathcal{L}(E)\cdot \langle P,P\rangle^{\mathrm{Sch}} \pmod{L^\times}.$$

Proof. The first part follows from the main result of [BD2], and the other two are [Ven, Theorems D and E]. We refer the reader to loc. cit. for a definition of the corresponding pairings. □

IV. Results beyond modular forms of weight 2. The main result of Bertolini and Darmon [BD2] was generalized by Seveso [Se] to modular forms of even weight. Let us recall here his main result for the sake of completeness and to illustrate that most of our results generalize to the situation of weights $(k,1,1)$, by replacing the points over the elliptic curve by the corresponding Heegner cycles. Let $f_k \in S_k(N)$, where $N = pN^+N^-$ and N^- is the squarefree product of an odd number of prime factors. The modular form corresponds, via the Jacquet-Langlands correspondence, to a modular form on a certain Shimura curve $X = X_{N^+,pN^-}$ uniformized by the p-adic upper half-plane. In this framework, Iovita and Spiess [IS] constructed a Chow motive $\mathcal{M}_{k-2}$ attached to modular forms on X. Let $m = k/2 - 1$.

We fix $K/\mathbb{Q}$ a quadratic imaginary field extension, of discriminant D_K prime to pN, such that N^+ is a product of primes that are split in K, while pN^- is a product of primes that are inert in K; we further fix an order of K of conductor c prime to ND_K. Hence, one may consider a higher weight analogue of Heegner points, the Heegner cycles $y_\psi^{(n)} \in \mathrm{CH}^{m+1}(\mathcal{M}_n)$ attached to a character ψ. The p-adic étale Abel-Jacobi map takes the form
$$\mathrm{AJ}_p : \mathrm{CH}^{m+1}(\mathcal{M}_n) \to M_k^\vee.$$

In the following result, the Mazur–Kitagawa p-adic L-function is replaced by the p-adic L-function attached to the quadratic imaginary field and the character ψ, that we denote by $\mathcal{L}(\mathbf{f}/K,\psi)(k,s)$ following the notations of [Se].

Proposition 2.4 (Seveso)**.** *The first derivative of $\mathcal{L}(f/K,\psi)(k,s)$ in the weight direction is given by*
$$2\frac{d}{dx}\Big(\mathcal{L}(\mathbf{f}/K,\psi)(x,x/2)\Big)\Big|_{x=k} = \mathrm{AJ}_p(y_\psi^{(n)})(f) + (-1)^m \mathrm{AJ}_p(y_{\bar\psi}^{(n)})(f).$$

This suggests that some of our results can be transposed to a higher weight situation, replacing the points over the elliptic curve by the corresponding Heegner cycles. More precisely, the results relying on the work of Darmon, Lauder and Rotger on the elliptic Stark conjecture [DLR1] can be adapted following the generalizations of Gatti and Guitart to higher weights [GG]. Similarly, the construction of *derived* cohomology classes, anticipated in the introduction and developed in Section 3, can be also carried out for general weights $(k,1,1)$.

3. Derived diagonal cycles and an explicit reciprocity law

3.1. **Diagonal cycles and an explicit reciprocity law.** Darmon and Rotger constructed in [DR3] an element
$$\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}}^\dagger)$$

arising from the interpolation of diagonal cycles along the balanced region (and depending on the choice of the triple of test vectors). An alternative construction has been given by Bertolini, Seveso and Venerucci [BSV1, Section 3]. This class is symmetric in all three variables. Let

$$\mathrm{res}_p : H^1(\mathbb{Q}, \mathbb{V}^{\dagger}_{\mathbf{fgh}}) \to H^1(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{fgh}})$$

denote the restriction map to the local cohomology at p, and set

$$\kappa_p(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathrm{res}_p(\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})) \in H^1(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{fgh}}).$$

One of the main results of both [BSV1] and [DR3] is the proof of an explicit reciprocity law. As showed in loc. cit., the Galois representation $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ is endowed with a four-step filtration

$$0 \subset \mathbb{V}^{++}_{\mathbf{fgh}} \subset \mathbb{V}^{+}_{\mathbf{fgh}} \subset \mathbb{V}^{-}_{\mathbf{fgh}} \subset \mathbb{V}^{\dagger}_{\mathbf{fgh}} \tag{11}$$

by $G_{\mathbb{Q}_p}$-stable $\Lambda_{\mathbf{fgh}}$-submodules of ranks 0, 1, 4, 7 and 8, respectively. Moreover,

$$\mathbb{V}^{+}_{\mathbf{fgh}}/\mathbb{V}^{++}_{\mathbf{fgh}} = \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}} \oplus \mathbb{V}^{\mathbf{hf}}_{\mathbf{g}} \oplus \mathbb{V}^{\mathbf{fg}}_{\mathbf{h}}.$$

We discuss now the definition of $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$. Let $\Theta^{\mathbf{gh}}_{\mathbf{f}}$ be the $\Lambda_{\mathbf{fgh}}$-adic cyclotomic character whose specialization at a point of weight (k, ℓ, m) is $\varepsilon^t_{\mathrm{cyc}}$, with $t := (-k + \ell + m)/2$, and let $\psi^{\mathbf{gh}}_{\mathbf{f}}$ be the unramified character of $G_{\mathbb{Q}_p}$ sending Fr_p to $\chi_f^{-1}(p)\mathbf{a}_p(\mathbf{f})\mathbf{a}_p(\mathbf{g})^{-1}\mathbf{a}_p(\mathbf{h})^{-1}$. Define $\mathbb{U}$ as the unramified $\Lambda_{\mathbf{fgh}}$-adic representation of $G_{\mathbb{Q}_p}$ given by the character $\psi^{\mathbf{gh}}_{\mathbf{f}}$, and let

$$\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}} = \mathbb{U}(\Theta^{\mathbf{gh}}_{\mathbf{f}}).$$

We finally introduce the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U}\hat{\otimes}\mathbb{Z}_p^{\mathrm{nr}})^{G_{\mathbb{Q}_p}}.$$

Then, one may construct a Perrin-Riou regulator map whose source is $H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}) \to \Lambda_{\mathbf{fgh}}$ and which interpolates either the Bloch–Kato logarithm or the dual exponential map, according to the value of a certain Hodge–Tate weight. In order to state their main properties, we need to introduce more terminology. Let $c = \frac{k+\ell+m-2}{2}$, and with the previous notations, define

$$\mathcal{E}^{\mathrm{PR}}(x, y, z) = \frac{1 - p^{-c}\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1 - p^{-c}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}.$$

The following result is discussed e.g. in [DR3, Proposition 5.6] and follows from the general theory of Perrin-Riou maps. For this statement we implicitly assume that neither $\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}$ nor $\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}$ vanish. If this were the case, and as we will later see, we need to work with the expression

$$\mathcal{E}^{\mathrm{PR}}(x, y, z) = \frac{1 - \chi_f(p)p^{\frac{k-\ell-m}{2}}\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1 - \bar{\chi}_f(p)p^{\frac{\ell+m-k-2}{2}}\alpha_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}^{-1}\alpha_{\mathbf{h}_z}^{-1}},$$

which agrees with the former in the non-exceptional case.

Proposition 3.1. *There is a homomorphism (usually named Perrin-Riou regulator)*

$$\mathcal{L}_{\mathbf{f},\mathbf{gh}} : H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}) \to \mathbb{D}(\mathbb{U})$$

such that for all $\kappa_p \in H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}})$ the image $\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p)$ satisfies the following interpolation properties:

(1) *For all points $(x, y, z) \notin \mathcal{W}^f_{\mathbf{fgh}}$,*

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p)) = \frac{(-1)^t}{t!}\mathcal{E}^{\mathrm{PR}}(x, y, z) \cdot \log_{\mathrm{BK}}(\nu_{x,y,z}(\kappa_p)),$$

(2) *For all points* $(x,y,z) \in \mathcal{W}^f_{\mathbf{fgh}}$,

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p)) = (-1)^t \cdot (1-t)! \cdot \mathcal{E}^{\mathrm{PR}}(x,y,z) \cdot \exp^*_{\mathrm{BK}}(\nu_{x,y,z}(\kappa_p)).$$

Following [DR3], one can define

$$\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^f \in H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}) \tag{12}$$

as the projection of the local class $\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$. Let $\eta_{\mathbf{f}}$, $\omega_{\mathbf{g}}$ and $\omega_{\mathbf{h}}$ be the canonical differentials attached to the three Hida families, as introduced for instance in [KLZ, Section 10]. The following result has been independently established in [BSV1, Theorem A] and [DR3, Theorem 5.1].

Proposition 3.2. *For any choice of test vectors, the following equality holds in the ring of fractions of* $\Lambda_{\mathbf{fgh}}$*:*

$$\langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^f), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle = \mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}).$$

Remark 3.3. Observe that both the left hand side and the right hand side depend on the choice of test vectors. There exist analogue reciprocity laws for $\mathscr{L}_p{}^g$ and $\mathscr{L}_p{}^h$.

We can also formulate an explicit reciprocity law for the improved p-adic L-function. Since along the region $k = \ell + m$ the Perrin-Riou map interpolates the dual exponential, we have that

$$\frac{1}{1-p^{-k+1}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}} \cdot \langle \exp^*_{\mathrm{BK}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^f(x,y,z)), \eta_{\mathbf{f}_x} \otimes \omega_{\mathbf{g}_y} \otimes \omega_{\mathbf{h}_z} \rangle = \mathcal{L}_p^f(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)^*, \tag{13}$$

and in particular the dual exponential map vanishes at (x_0,y_0,y_0) (i.e. the class is crystalline) if and only if the improved p-adic L-function is zero at that point.

Remark 3.4. In [GGMR], the authors study the cohomology classes in a generic rank zero situation, where they are non-crystalline. This yields a formula for the special value $\mathscr{L}_p{}^g$ in terms of $\mathscr{L}_p{}^f$ in absence of exceptional zeros. Again, the key point is that each component of the cohomology class encodes information about a different p-adic L-function.

3.2. **Vanishing of cohomology classes.** In [BSV1, Section 9], the authors deal with a situation where the numerator of the Perrin-Riou map $\mathcal{L}_{\mathbf{f},\mathbf{gh}}$ vanishes, defining an improved map whose derivatives may be explicitly computed. We come back to this question later on. Let us analyze, firstly, the vanishing of the denominator of the Perrin-Riou map, but in the case of the Perrin-Riou map $\mathcal{L}_{\mathbf{g},\mathbf{hf}}$, that is:

$$1 - \bar{\chi}_g(p)p^{\frac{k-\ell+m-2}{2}}\alpha_{\mathbf{f}_x}^{-1}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}^{-1} = 0. \tag{14}$$

Since we have placed ourselves in the ordinary setting, a necessary condition for this to happen is $k+m = \ell+2$, which moreover suffices to guarantee the analyticity of the Euler factor in the denominator.

Hence, when f is of weight 2 with split multiplicative reduction at p, and g and h are self-dual of the same weight ($h = g \otimes \chi_g^{-1}$), the denominator of the Perrin-Riou map vanishes. This means that we expect

$$\log_{\mathrm{BK}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^g(x,y,z)) = 0.$$

However, the self-duality condition is not necessary for this vanishing, and following the conventions of the introduction in the case of weights $(2,1,1)$ (again, with E of split multiplicative reduction), it suffices to impose that $\alpha_g\beta_h = 1$. This encompasses for example the case of theta series of quadratic fields where the prime p is inert. Nevertheless, there are certain phenomena which are exclusive from the self-dual case: indeed, the fact that the Hida families interpolating both g and h keep the self-duality condition gives us a vanishing along the whole line $(2, \ell, \ell)$. We treat both the self-dual and the non self-dual case, emphasizing the main differences between them.

We begin by showing that when $\alpha_g \beta_h = 1$ and g and h are self-dual, the local class $\kappa_p(f, g_\alpha, h_\alpha)$ vanishes, using the techniques of our prior work [RiRo1]. Although this is not strictly necessary since we will later see that the *whole* global class is zero, we believe that it may be instructive for the reader to compare the formalism of [RiRo1], which relies on the basic properties of the Perrin-Riou maps, with the more conceptual proof of [BSV1, Section 9], based on the geometric construction of an *improved* cohomology class.

Proposition 3.5. *With the running assumptions, the specialization of the Λ-adic cohomology class $\kappa_p(\mathbf{f}, \mathbf{g}, \mathbf{g}^*)$ at (x_0, y_0, y_0) vanishes, that is, $\kappa_p(f, g_\alpha, g^*_{1/\beta}) = 0$.*

Proof. We will follow the same strategy used in [RiRo1, Theorem 3.5]. First of all we show, invoking [BSV1, Theorem 8.1], that any specialization of the three-variable Λ-adic class at a point of weights $(2, \ell, \ell)$, with $\ell \geq 2$, is zero. In order to achieve this, we just use the comparison provided by the aforementioned result with the twisted class $\kappa^\dagger$, twisting now in the g-variable, that is, applying the operator $\mathrm{Id} \otimes w_p' \otimes \mathrm{Id}$ according to the definitions given at the beginning of Section 7.2 of loc. cit., where w_p stands for the Atkin–Lehner involution. As we later discuss, this class may be understood as an *improved* cohomology class, since it agrees with the former up to multiplication by the Euler factor

$$1 - \frac{\bar{\chi}(p)\alpha_{\mathbf{g}_y}}{\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}} p^{\frac{k-\ell+m-2}{2}}.$$

This factor is zero over the line $(2, \ell, \ell)$ when we take Hida families such that $\mathbf{h} = \mathbf{g}^*$, since $\bar{\chi}(p)\alpha_{\mathbf{g}_y} = \alpha_{\mathbf{h}_y}$. Observe that we are implicitly using Lemma 8.4 of loc. cit., which assert that the class $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is symmetric in all three variables.

The second part of the proof consists on applying a limit argument componentwise, via the corresponding Perrin-Riou maps, to conclude that the limit when ℓ goes to one is also zero. For this last step, we look at the four different components of the local class $\kappa_p(f, g_\alpha, g^*_{1/\beta})$ corresponding to the *balanced* subspace $\mathbb{V}^+_{fgg^*}$. This suffices according to the results established in [BSV1, Corollary 8.2] and following the notations of Section 6.2 in loc. cit., which asserts that the three-variable cohomology class lies in the balanced subspace. The components of the balanced subspace are denoted by $V_f^{gg^*}$, $V_g^{g^*f}$, $V_{g^*}^{fg}$ and $V^{++}_{fgg^*}$, where $V_f^{gg^*}$ stands for the specialization of $\mathbb{V}_{\mathbf{f}}^{\mathbf{gg}^*}$ and similarly for the other factors (recall the filtration of (11)).

- We first prove that the component associated to the rank one subspace $V_f^{gg^*}$ is zero. Observe that along the line $(2, \ell, \ell)$, the specialization of the module $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gg}^*})$ agrees with $H^1(\mathbb{Q}_p, \mathbb{Z}_p(\psi_{g_y}^{-2})(\ell - 1))$, where y is a point of weight ℓ. Then, the Perrin-Riou map is an application

$$H^1(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-2}) \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}})) \to \mathbb{D}(\Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-2})) \hat{\otimes} \Lambda. \tag{15}$$

  Since $\psi_{\mathbf{g}}^{-2} \neq 1$, we have $H^0(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-2})) = 0$ and it follows from [KLZ, Theorem 8.2.3] that the above map is an isomorphism. Moreover, using the same argument of the proof of the last step of [RiRo1, Theorem 3.5], we conclude that the Λ-module of (15) is non-canonically isomorphic to $\Lambda_{\mathbf{g}}$. Therefore, and since infinitely many specializations vanish according to the previously quoted result of [BSV1], the corresponding H^1 is zero.
- The components associated to $V_g^{g^*f}$ and $V_{g^*}^{fg}$ are zero; this is because

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{\mathbf{g}^*\mathbf{f}}|_{(2,\ell,\ell)}) \simeq H^1(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(1)) \simeq \Lambda_{\mathbf{g}} \oplus \Lambda_{\mathbf{g}},$$

  and although the Perrin-Riou map only kills one of the above two components, the restriction of the class is zero since again infinitely many specializations are zero.

- For the remaining component, the one corresponding to $\mathbb{V}_{fgg^*}^{++}$, the same argument used in the first step works once we have established that the remaining projections vanish.

$\square$

Consider now the surface

$$\mathcal{S} = \mathcal{S}_{k,k+m-2,m} := \{(x,y,z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} : \mathrm{wt}(x) + \mathrm{wt}(z) = \mathrm{wt}(y) + 2\},$$

and also the line

$$\mathcal{C} := \{(x,y,z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} : \mathrm{wt}(x) = 2, \quad \mathrm{wt}(y) = \mathrm{wt}(z)\}.$$

Observe that the surface $\mathcal{S}$ is just a finite cover of the plane in $\mathcal{W}^3$ arising as the Zariski closure of weights $(k, k+m-2, m)$.

Using the results of [BSV1, Section 9.2], we may upgrade Proposition 3.5 to the vanishing of the global class $\kappa(f, g_\alpha, h_\alpha)$ when $\alpha_g \beta_h = 1$ (and hence we can work beyond the setting of the adjoint, covering for example the case of theta series of quadratic fields where the prime p remains inert).

In particular, we have the following result.

Proposition 3.6. *The global class $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ vanishes along the line $\mathcal{C}$ in the self dual case. Moreover, the class $\kappa(f, g_\alpha, h_{1/\beta})$ is zero when $\alpha_g \beta_h = 1$.*

Proof. Following again [BSV1, Section 9.2], there is an *improved* class $\kappa_g^*(\mathbf{f}, \mathbf{g}, \mathbf{h})$ along the surface $\mathcal{S}$ satisfying

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})_{|\mathcal{S}} = \Big(1 - \frac{\bar{\chi}(p)\alpha_{\mathbf{g}_y}}{\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}}\Big)\kappa_g^*(\mathbf{f}, \mathbf{g}, \mathbf{h}). \tag{16}$$

Hence, the vanishing of $\kappa(f, g_\alpha, h_{1/\beta})$ follows from the vanishing of the corresponding Euler factor. $\square$

3.3. **Derived classes and reciprocity laws.** Following the analogy with [RiRo1], let us focus firstly on the self-dual case to discuss the notion of *derived* classes. We shrink the weight space $\mathcal{W}$ to a rigid-analytic open disk $\mathcal{U} \subset \mathcal{W}$ centered at 2 at which the finite cover $w : \mathcal{W}_{\mathbf{f}} \to \mathcal{W}$ restricts to an isomorphism $w : \mathcal{U}_{\mathbf{f}} \xrightarrow{\sim} \mathcal{U}$ with $x_0 \in \mathcal{U}_{\mathbf{f}}$. Let $\Lambda_{\mathcal{U}_{\mathbf{f}}} = \mathcal{O}(\mathcal{U}_{\mathbf{f}})$ denote the Iwasawa algebra of analytic functions on $\mathcal{U}_{\mathbf{f}}$ whose supremum norm is bounded by 1. Shrink likewise $\mathcal{C}$ and $\mathcal{S}$ so that projection to the weight space restricts to an isomorphism with $\mathcal{U}$ and $\mathcal{U} \times \mathcal{U}$ respectively. Having done that, their associated Iwasawa algebras are respectively $\mathcal{O}(\mathcal{C}) = \Lambda_{\mathcal{U}_{\mathbf{f}}} \simeq \mathbb{Z}_p[[X]]$ and $\mathcal{O}(\mathcal{S}) = \Lambda_{\mathcal{U}_{\mathbf{f}}} \hat{\otimes} \Lambda_{\mathcal{U}_{\mathbf{h}}} \simeq \mathbb{Z}_p[[X, Z]]$. The isomorphism $\Lambda_{\mathcal{U}_{\mathbf{f}}} \simeq \mathbb{Z}_p[[X]]$ is not canonical and depends on the choice of an element $\gamma \in \Lambda_{\mathcal{U}_{\mathbf{f}}}^\times$ which is sent to $1 + X$.

Then, consider the short exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \mathbb{Z}_p[[X, Z]] \xrightarrow{\cdot X} \mathbb{Z}_p[[X, Z]] \to \mathbb{Z}_p[[Z]] \to 0.$$

Under the usual isomorphisms, $\Lambda_{\mathbf{f}}$ may be identified with $\mathbb{Z}_p[[X]]$ after fixing a topological generator γ of $\Lambda_{\mathcal{U}_{\mathbf{f}}}^\times$ and sending $[\gamma]$ to $1 + X$. Then, $\Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{h}}$ becomes isomorphic to $\mathbb{Z}_p[[X, Z]]$ and the previous exact sequence may be recast as

$$0 \to \mathcal{O}_{\mathcal{S}} \xrightarrow{\delta} \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{C}} \to 0 \tag{17}$$

with $\delta = (\gamma - 1) \otimes 1$ in $\mathcal{O}_{\mathcal{S}} \simeq \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{h}}$.

Proposition 3.7. *In the self-dual case, there exists a class $\kappa_\gamma'(\mathbf{f}, \mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgg}^*|\mathcal{S}})$ such that*

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{g}^*)_{|\mathcal{S}} = \delta \cdot \kappa_\gamma'(\mathbf{f}, \mathbf{g}, \mathbf{g}^*).$$

If we further assume that $\mathbb{V}_{\mathbf{fgg}^*|\mathcal{C}}^{G_{\mathbb{Q}}} = \{0\}$*, the class is unique.*

Proof. This follows by considering the long exact sequence in cohomology attached to (17):

$$H^0(\mathbb{Q}, \mathbb{V}_{\mathbf{fgg}^*|\mathcal{C}}) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgg}^*|\mathcal{S}}) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgg}^*|\mathcal{S}}) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgg}^*|\mathcal{C}}).$$

Since the restriction of $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{g}^*)$ to $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgg}^*|\mathcal{C}})$ is zero by Proposition 3.6, one may assure the existence of a *derived* class as in the statement. □

Note that over $\mathbb{Q}_p$, the uniqueness of a derived class with values in the balanced subspace follows by an analysis of the Hodge–Tate weights.

Remark 3.8. Normalizing by $\log_p(\gamma)$, the specializations of this class over the line $(2, \ell, \ell)$ can be proved to be independent of the choice of γ.

In general, if we are no longer in the self-dual case but the condition $\alpha_g \beta_h = 1$ still holds, the notion of derived class makes sense at the point (x_0, y_0, z_0). For that purpose, let $\mathcal{D}$ stand for the codimension two subvariety

$$\mathcal{D} := \{(x, y, z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} : \mathrm{wt}(x) = \mathrm{wt}(y) + 1, \quad z = z_0\}.$$

The following result is the analogue of [RiRo1, Proposition 3.13] and its proof follows from the same argument of Proposition 3.7.

Proposition 3.9. *Assume that* $\alpha_g \beta_h = 1$ *and that* $V_{fgh}^{G_{\mathbb{Q}}} = \{0\}$*. Then, there exists a unique class* $\kappa'_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}|\mathcal{D}})$ *such that*

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})|_{\mathcal{D}} = \delta \cdot \kappa'_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{h}).$$

Let $\mathcal{L} = \frac{\alpha'_g}{\alpha_g} - \frac{\alpha'_f}{\alpha_f}$, and consider the normalization of $\kappa'_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{h})$ by γ, that is,

$$\kappa'(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \frac{\kappa'_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{h})}{\log_p(\gamma)}.$$

The following result can be seen as an improved reciprocity law expressing the logarithm of the derived local class in terms of a p-adic L-value.

Theorem 3.10. *The logarithm of the derived local class satisfies the following*

$$\langle \log_{\mathrm{BK}}(\kappa'_p(\mathbf{f}, \mathbf{g}, \mathbf{h})^g(x_0, y_0, z_0)), \omega_f \otimes \eta_g \otimes \omega_h \rangle = \mathcal{L} \cdot \mathscr{L}_p{}^{g_\alpha}(\mathbf{f}, \mathbf{g}, \mathbf{h})(x_0, y_0, z_0) \pmod{L^\times},$$

where as in (12) *the superindex* g *refers to the projection to* $\mathbb{V}_{\mathbf{g}}^{\mathbf{hf}}$.

Proof. Consider the reciprocity law of Proposition 3.2, now for $\mathscr{L}_p{}^{g_\alpha}(\mathbf{f}, \mathbf{g}, \mathbf{h})$, restricted to $\mathcal{D}$, and multiply both sides by the Euler factor in the denominator of the Perrin-Riou map. Then, we have an equality of the form

$$\Big(1 - \frac{\bar\chi(p)\alpha_{\mathbf{g}_y}}{p\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}}\Big) \cdot \langle \log_{\mathrm{BK}}(\kappa_p(\mathbf{f}, \mathbf{g}, \mathbf{h})^g), \omega_{\mathbf{f}} \otimes \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle = \Big(1 - \frac{\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}}{\bar\chi(p)\alpha_{\mathbf{g}_y}}\Big) \cdot \mathscr{L}_p{}^{g_\alpha}(\mathbf{f}, \mathbf{g}, \mathbf{h}),$$

since along $\mathcal{D}$ the Perrin-Riou interpolates the Bloch–Kato logarithm. At the point (x_0, y_0, z_0) both the cohomology class at the left hand side and the Euler factor at the right are zero. Taking derivatives along the direction $(k+1, k, 1)$, and evaluating then at the point (x_0, y_0, z_0), the result follows (see [Ri1, Remark 4.8] for a more exhaustive discussion on the identifications we are considering). □

An analogue formula holds for any point over the line $(2, \ell, \ell)$ in the self-dual case, but of course the description of the $\mathcal{L}$-invariant is not so explicit and relies on the results of [Se].

It may be instructive to compare this *derived* cohomology class with the *improved* cohomology class considered by Bertolini, Seveso and Venerucci. We can prove the following.

Proposition 3.11. *Consider the map given by the projection*

$$\phi_g^{hf} : H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}|\mathcal{S}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{\mathbf{hf}}|_{\mathcal{S}}).$$

Then, there is a relation between the improved class $\phi_g^{hf}(\kappa_g^*(\mathbf{f},\mathbf{g},\mathbf{h}))$ *and* $\phi_g^{hf}(\kappa'(\mathbf{f},\mathbf{g},\mathbf{h}))$*, given by*

$$\phi_g^{hf}(\kappa'(\mathbf{f},\mathbf{g},\mathbf{h})) = \mathcal{L} \cdot \phi_g^{hf}(\kappa_g^*(\mathbf{f},\mathbf{g},\mathbf{h})) \pmod{L^\times}.$$

Proof. This is proved by applying the map $\langle \log_{\mathrm{BK}}(\cdot), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle$ to both sides, and then comparing the results. For that purpose, we use that the Euler factors involved in the Perrin-Riou map are analytic along $\mathcal{S}$ and can be cancelled out. That way, we obtain an *improved* reciprocity law

$$\mathscr{L}_p^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = \langle \log_{\mathrm{BK}}(\phi_g^{hf}(\kappa_g^*(\mathbf{f},\mathbf{g},\mathbf{h})^g(x,y,z))), \omega_{\mathbf{f}_x} \otimes \eta_{\mathbf{g}_y} \otimes \omega_{\mathbf{h}_z} \rangle \pmod{L^\times},$$

which holds for all the points (x,y,z) of $\mathcal{S}$. □

Finally, we point out that we may expect a relation between $\kappa_p'(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z)$ and the Gross–Kudla–Schoen cycle of [DR1], that we denote by $\Delta_{k,\ell,m} \in H^1(\mathbb{Q}, V_{fgh}((4-k-\ell-m)/2)$. In particular, we expect the following result to be true (or at least, a slight variant of it). Here, loc_p stands for the localization at p-map.

Question 3.12. Can we establish that, up to multiplication by a non-zero constant in $L^\times$ and for any point (x,y,z) of weights $(2,\ell,\ell)$ with $\ell \geq 2$, we have the equality

$$\kappa_p'(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = \mathcal{L} \cdot \mathrm{loc}_p(\Delta_{2,\ell,\ell})?$$

Of course, this would require the proof of an analogue result to [DR1, Theorem 5.1] in a situation where f has split multiplicative reduction.

4. Derivatives of triple product p-adic L-functions

In this section, we discuss a variant of the elliptic Stark conjecture for the derivative of the triple product p-adic L-function $\mathscr{L}_p^f$ in a situation of exceptional zeros. As before, we keep the assumption that f has split multiplicative reduction at p and that an exceptional zero condition occurs.

There are two main instances we want to consider: the rank zero situation and the rank two situation. While the former is quite well understood after the results developed in [BSV1] and [BSV2], the latter is more subtle and we will propose a conjectural formula in this scenario. Along this section, by the word *rank*, we refer to the rank of the V_{gh}-isotypic component of $E(H)$. According to our general assumptions on the local signs, the rank is always even. The V_{gh}-component of $E(H)$ is endowed with an inclusion in the Selmer group, that is,

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(E(H), V_{gh}) \simeq (E(H) \otimes V_{gh})^{G_{\mathbb{Q}}} \subset H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh}),$$

where $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$ is the group of extensions of $\mathbb{Q}_p$ by V_{fgh} in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$ which are crystalline at p.

Recall that for higher ranks the computations performed in [DLR1] lead us to expect that the special value $\mathscr{L}_p^{g_\alpha}$ presented in the introduction is zero, and that the second derivative of $\mathscr{L}_p^f$ along the f-direction vanishes, too. The odd rank situation is equally interesting, and we hope to come back to this question in a further work. We keep the notations of the previous section.

Proposition 4.1. *The value* $\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(x_0,y_0,z_0)$ *is zero. Moreover, the jacobian matrix of* $\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})$ *at the point* (x_0,y_0,z_0) *is given by*

$$(0 \quad \mathcal{L}_{g_\alpha} - \mathcal{L}_f \quad \mathcal{L}_{h_\alpha} - \mathcal{L}_f) \cdot \mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})^*.$$

Proof. This directly follows from [BSV1, Proposition 9.2]. □

Remark 4.2. Observe that, towards the rationality conjectures we are interested in, the value ${\mathscr{L}_p}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$ is an algebraic number, and it is non-zero if and only if the cohomology class $\kappa(f,g_\alpha,h_\alpha)$ is non-crystalline.

In particular, the derivative along the direction $(2+k,1,1)$ vanishes, and along the direction $(2,1+\ell,1+\ell)$ is given by $\mathcal{L}_{g_\alpha}+\mathcal{L}_{h_\alpha}-2\mathcal{L}_f$, up to an explicit algebraic number in the number field L.

Suppose from now on that ${\mathscr{L}_p}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$ vanishes at (x_0,y_0,z_0). Therefore, the cohomology class $\kappa(f,g_\alpha,h_\alpha)$ is crystalline, and following [BSV2, Section 2.1] we may define a *new* Bloch–Kato logarithm, denoted by $\log_{\beta\beta}$ in loc. cit. Roughly speaking, it can be understood as a projection to the rank one subspace V_{fgh}^{++} arising in the filtration (11), followed by the Bloch–Kato logarithm and the pairing with $\omega_f\otimes\omega_g\otimes\omega_h$. To be coherent with the other notations we will need later on, write $\log^{++}$ for this map. Alternatively, we may consider the local class $\kappa_p(f,g_\alpha,h_\alpha)$ and take its decomposition according to the action of the Frobenius element, in such a way that $\kappa_{\beta\beta}$ is the part corresponding to the (β_g,β_h) component.

Assume further that $\alpha_g\alpha_h=1$ (in particular, this also implies that $\beta_g\beta_h=1$). The following result is the content of [BSV2, Section 2.1].

Proposition 4.3. *Under the given conditions, the value ${\mathscr{L}_p}^f(\mathbf{f},g_\alpha,h_\alpha)$ vanishes and*

$$\frac{d^2}{dx^2}{\mathscr{L}_p}^f(\mathbf{f},g_\alpha,h_\alpha)|_{x=x_0}=\frac{1}{2\mathrm{ord}_p(q_E)}\cdot(1-p^{-1})^{-1}\cdot\log^{++}(\kappa_p(f,g_\alpha,h_\alpha)).$$

Remark 4.4. In the adjoint case, when we take $h_{1/\alpha}=g^*_{1/\beta}$ we do have a relation between $\mathcal{L}_g$ and $\mathcal{L}_h$: indeed

$$\frac{(1/\alpha_g)'}{1/\alpha_g}=-\frac{\alpha_g'}{\alpha_g};$$

however when $h_{1/\alpha}=g^*_{1/\alpha}$ both quantities are a priori unrelated.

4.1. **A conjecture for the second derivative.** As we have discussed before, the improved p-adic L-function ${\mathscr{L}_p}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$ interpolates an explicit non-zero multiple of $L(f\otimes g\otimes h,1)$, and we expect this value to be zero when the rank of the corresponding isotypic component of the Selmer group is two. In those cases, we would like to compare the Kato class with a basis of $(E(H)\otimes V_{gh})^{G_{\mathbb{Q}}}$, that we write as $\{P,Q\}$. We also assume that $H^1_{\mathrm{f}}(\mathbb{Q},V_{fgh})$ has dimension 2.

To fix notations, observe that V_{gh} decomposes as a $G_{\mathbb{Q}_p}$-module as the direct sum of four different lines $V_{gh}^{\alpha\alpha}:=V_g^{\alpha_g}\otimes V_h^{\alpha_h},\dots,V_{gh}^{\beta\beta}$. After choosing a basis of V_{gh}, we may write this decomposition as

$$V_{gh}=L\cdot e_{\alpha\alpha}\oplus L\cdot e_{\alpha\beta}\oplus L\cdot e_{\beta\alpha}\oplus L\cdot e_{\beta\beta},$$

where

$$\mathrm{Fr}_p(e_{\lambda\mu})=a_{\lambda\mu}\cdot e_{\lambda\mu}\quad\text{ for any }\quad\lambda,\mu\in\{\alpha,\beta\}.$$

Here, $a_{\lambda\mu}=\beta_g\beta_h$ if $(\lambda,\mu)=(\alpha,\alpha)$ and similarly in the other three cases.

In particular, restricting the elements $\{P,Q\}$ to a decomposition group at p gives expressions

$$P=P_{\beta\beta}\otimes e_{\beta\beta}+P_{\beta\alpha}\otimes e_{\beta\alpha}+P_{\alpha\beta}\otimes e_{\alpha\beta}+P_{\alpha\alpha}\otimes e_{\alpha\alpha},$$

and similarly for Q, where as recalled in the introduction Fr_p acts on $P_{\beta\beta}$ with eigenvalue $\beta_g\beta_h$ and analogously for the remaining components.

Conjecture 4.5. *Under the running assumptions, the following equality holds:*

$$\frac{d^2}{dx^2}{\mathscr{L}_p}^f(\mathbf{f},g_\alpha,h_\alpha)|_{x=x_0}=\log_p(P_{\beta\beta})\log_p(Q_{\alpha\alpha})-\log_p(Q_{\beta\beta})\log_p(P_{\alpha\alpha})\pmod{L^\times}.$$

This conjecture can be seen as a quite natural analogue for the first part of Proposition 2.3; that is, we are proposing an expression for the second derivative of the p-adic L-function along the line $(k, k/2)$ since the central critical point corresponding to $(k, 1, 1)$ is precisely $k/2$. It would be interesting to understand the derivatives along different directions; we expect that they would be related with appropriate height pairings. See [CaHs] for an approximation to that question when g and h are theta series attached to the same quadratic imaginary field where the prime p splits.

4.2. **Some reducible cases.** We continue by recalling some factorization formulas in special cases where the representation V_{gh} becomes reducible. See [DLR2, Section 2] for a complete discussion of the different cases where this may occur.

A first case occurs when g and h are theta series of the same quadratic field K, but the behavior is ostensibly different depending on whether K is real or imaginary, and on whether p is inert or split in K. While the inert case was worked out in [BSV2], the split case was not considered in loc. cit. However, it turns out that it is not specially interesting, at least when K is imaginary: the second derivative of $\mathscr{L}_p{}^f$ along the x-direction is 0 for trivial reasons.

Remark 4.6. It may be tempting to prove a factorization formula for $\mathscr{L}_p{}^f$ as in [CR], or even when all three variables (k, ℓ, m) are allowed to move along a Hida family. However, the two-variable Castella's p-adic L-functions considered in loc. cit. would have infinity types

$$\Big(\frac{k+\ell+m}{2}-1, \frac{k+\ell+m}{2}-\ell-m+1\Big), \quad \Big(\frac{k+\ell+m}{2}-m, \frac{k+\ell+m}{2}-\ell\Big).$$

This precludes the possibility of comparing the different p-adic L-values along the region of classical interpolation, since they are disjoint.

Finally, in the case where $h = g^*$, the situation is also quite simple and the right hand of the conjecture is zero when the component corresponding to the adjoint has rank two. For details on that, see the case by case analysis, completely analogue to our situation, of [DR2].

Proposition 4.7. *Conjecture 4.5 holds whenever (a) g and h are theta series of an imaginary quadratic field where p splits, $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, with each component of rank one; (b) g and h are theta series of a quadratic field where p is inert, $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, and either ψ_1 or ψ_2 is a genus character.*

Proof. Consider first the case of imaginary quadratic fields, where we can prove that both the left and the right hand side are zero for trivial reasons. For that purpose, recall the notations introduced in the discussion before Proposition 4.3. In order to see that the second derivative vanishes, it is enough to conclude that the component $\kappa_{\beta\beta} = 0$, and this follows after adapting the results of [DR2, Section 4.3] to the multiplicative situation, where one may invoke the discussion of [CR]. In particular, if we assume without loss of generality that $P_{\beta\alpha} \neq 0$, then $P_{\alpha\alpha} = P_{\beta\beta} = 0$, and similarly $Q_{\alpha\beta} = Q_{\beta\alpha} = 0$. See [GGMR, Section 4] for a similar treatment of an analogue situation.

The case of theta series for quadratic fields where the prime is inert follows from the main results of Bertolini–Seveso–Venerucci [BSV2, Section 3], taking into account the identifications among the different eigenspaces for the Frobenius action of e.g. [DLR1, Section 3.3] and [DR2, Sections 4.3, 4.4]. $\square$

4.3. **The conjecture in other settings.** We would like to make some comments regarding the case $\alpha_g\beta_h = 1$. Observe that the previous Euler factor that gave rise to the improved p-adic L-function does not vanish, but the factors

$$1 - \frac{\chi_f(p)\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}{\alpha_{\mathbf{f}_x}} p^{\frac{k-\ell-m}{2}} \quad \text{and} \quad 1 - \frac{\chi_f(p)\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{\alpha_{\mathbf{f}_x}} p^{\frac{k-\ell-m}{2}}$$

do. The first one is analytic along the region $\mathcal{S}_{k+m=\ell+2}$, while the second is analytic along the region $\mathcal{S}_{k+\ell=m+2}$. In this case we cannot assure the existence of an improved p-adic L-function, but at least we can guarantee that $\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})$ vanishes.

We get indeed a very similar result.

Proposition 4.8. *The value* $\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(x_0,y_0,z_0)=0$. *Moreover, the jacobian matrix of* $\mathscr{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})$ *at* (x_0,y_0,z_0) *is given by an* L*-multiple of*

$$(0 \quad \mathcal{L}_{g_\alpha}-\mathcal{L}_f \quad \mathcal{L}_{h_\alpha}-\mathcal{L}_f).$$

Observe that Conjecture 4.5 still makes sense in this framework. And again, we can also take the derivative along the line $(2+k,1+k,1)$ and we would expect to relate it with an explicit multiple of an appropriate height pairing $\langle P,P\rangle$.

5. Applications to the elliptic Stark conjecture

5.1. **Interplay between both settings and a conjecture of Darmon–Rotger.** Let H denote the smallest number field cut out by the representation V_{gh}, with coefficients in a finite extension $L/\mathbb{Q}$. By enlarging it if necessary, assume throughout that L contains both the Fourier coefficients of g and h, and the roots of their p-th Hecke polynomials. Fix a prime ideal $\wp$ of H lying above p, thus determining an embedding $H\subset H_p\subset\bar{\mathbb{Q}}_p$ of H into its completion H_p at $\wp$, and an arithmetic Frobenius $\mathrm{Fr}_p\in\mathrm{Gal}\,(H_p/\mathbb{Q}_p)$. Due to our regularity assumptions, V_g and V_h decompose as

$$V_g=V_g^\alpha\oplus V_g^\beta,\quad V_h=V_h^\alpha\oplus V_h^\beta,$$

where Fr_p acts on V_g^α with eigenvalue α_g, and similarly for the remaining summands.

Fix eigenbases $\{e_g^\alpha,e_g^\beta\}$ and $\{e_h^\alpha,e_h^\beta\}$ of V_g and V_h, respectively, which are compatible with the choice of the basis for V_{gh}, i.e.,

$$e_{\alpha\alpha}=e_g^\alpha\otimes e_h^\alpha,\quad e_{\alpha\beta}=e_g^\alpha\otimes e_h^\beta,\quad e_{\beta\alpha}=e_g^\beta\otimes e_h^\alpha,\quad e_{\beta\beta}=e_g^\beta\otimes e_h^\beta$$

(recall that in previous sections we were using the dual basis). Let $\eta_{g_\alpha}\in(H_p\otimes V_g^\beta)^{G_{\mathbb{Q}_p}}$ and $\omega_{h_\alpha}\in(H_p\otimes V_g^\alpha)^{G_{\mathbb{Q}_p}}$ denote the canonical periods arising as the weight one specializations of the Λ-adic periods $\eta_{\mathbf{g}}$ and $\omega_{\mathbf{h}}$ introduced in [KLZ, Section 10.1]. Then, we can define p-adic periods $\Xi_{g_\alpha}\in H_p^{\mathrm{Fr}_p=\beta_g^{-1}}$ and $\Omega_{h_\alpha}\in H_p^{\mathrm{Fr}_p=\alpha_h^{-1}}$ by setting

$$\Xi_{g_\alpha}\otimes e_g^\beta=\eta_{g_\alpha},\quad \Omega_{h_\alpha}\otimes e_h^\alpha=\omega_{h_\alpha},$$

and

$$\mathcal{L}_{g_\alpha}:=\frac{\Omega_{g_\alpha}}{\Xi_{g_\alpha}}\in(H_p)^{\mathrm{Fr}_p=\frac{\beta_g}{\alpha_g}}. \tag{18}$$

At the same time, recall that u_{g_α} is the Stark unit attached to the adjoint representation of g_α, which arises as a normalization term in the conjectures of [DLR1] and [DLR2] involving a *second-order* regulator. Then, it was conjectured by Darmon and Rotger [DR2] that

$$\mathcal{L}_{g_\alpha}=\log_p(u_{g_\alpha})\pmod{L^\times}. \tag{19}$$

This relation gives a relatively easy interpretation of the apparently mysterious unit u_{g_α}. This suggests that more natural descriptions of this object should be available, involving only the arithmetic of the modular form g. However, this conjecture seems to be hard to prove, even in cases where the elliptic Stark conjecture is known (theta series of quadratic imaginary fields where the prime p splits). The main difficulty is the lack of an explicit description of the periods Ω_{g_α} and Ξ_{g_α}: in weights greater than one, these periods can be understood as certain algebraic numbers and be explicitly described, but in weight one this description is no longer available and Ω_{g_α} and Ξ_{g_α} are p-adic transcendental numbers.

The main point of this section is that the knowledge of different conjectures involving these periods can be enough to determine the value of the ratio $\mathcal{L}_{g_\alpha}$. Indeed, the generalized cohomology classes described in Section 3 can be decomposed as the sum of different components, each one encoding information about different p-adic L-functions. When combining these results, we may relate the different periods which are involved.

As a first application of this technique, let us prove a result of this kind using the theory of Beilinson–Flach elements. This corresponds to the limit case where the modular form f is Eisenstein and the arithmetic governing the triple product are ostensibly different. For the following discussion, the notations are the same of [RiRo2]. Let $U_{gg^*} = \mathcal{O}_H^\times \otimes L$ and $U_{gg^*}[1/p] = \mathcal{O}_H[1/p]^\times \otimes L$, and assume that the hypothesis (H1)-(H3) of the introduction of [RiRo2] hold. Fix a basis $\{u, v\}$ of the two dimensional space $(U_{gg^*}[1/p]/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(V_g))^{G_\mathbb{Q}}$ such that $u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(V_g))^{G_\mathbb{Q}}$. As in the case of elliptic curves, these unit groups are endowed with a Frobenius action, since the restriction to a decomposition group allows us to decompose $\mathrm{ad}^0(V_g) = L^1 \oplus L^{\alpha/\beta} \oplus L^{\beta/\alpha}$ and we may take the projection of u and v to each of those components. Let

$$\begin{array}{rcl} R_{g_\alpha} & = & \log_p(u_1)\log_p(v_{\alpha/\beta}) - \log_p(v_1)\log_p(u_{\alpha/\beta}), \\ R_{g_\beta} & = & \log_p(u_1)\log_p(v_{\beta/\alpha}) - \log_p(v_1)\log_p(u_{\beta/\alpha}) \end{array}$$

be the regulators which appear in the formulation of the main conjecture of [DLR2] and [RiRo2].

Proposition 5.1. *Assume that R_{g_α} and R_{g_β} are both non-zero. Then,*

$$\frac{\mathcal{L}_{g_\alpha}}{\mathcal{L}_{g_\beta}} = \frac{\log_p(u_{g_\alpha})}{\log_p(u_{g_\beta})} \pmod{L^\times}.$$

Proof. Recall the maps $\log^{+-}$ and $\log^{-+}$ introduced in [RiRo2, Section 3.3] as the composition of the corresponding projection maps from V_{gh}, the Bloch–Kato logarithm, and the pairing with the canonical differentials. Apply then [RiRo2, Proposition 4.3] twice, first with the map $\log^{-+}$ (and hence taking the β/α component of both u and v), and then with the map $\log^{+-}$ (taking the α/β component of both u and v). Then, comparing both expressions we have that

$$\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}} \cdot \log_p(u_{g_\alpha}) = \Omega_{g_\alpha} \cdot \Xi_{g^*_{1/\alpha}} \cdot \log_p(u_{g_\beta}) \pmod{L^\times}.$$

We now proceed as in [RiRo1, Section 5.2] (see the discussion after display (56)), observing that

$$\Omega_{g^*_{1/\alpha}} = \Xi_{g_\beta}^{-1}, \quad \Xi_{g^*_{1/\alpha}} = \Omega_{g_\beta}^{-1} \pmod{L^\times},$$

and we are done. □

We would like to go a step beyond and aim for stronger results, so in a certain way we would like to keep the period attached to h fixed and vary just the one attached to g, which would yield the desired equality.

We do this by analyzing first the prototypical case of the elliptic Stark conjecture, where the Selmer group is two-dimensional and we may fix a basis $\{P, Q\}$ of the L-vector space

$$(E(H) \otimes V_{gh})^{G_\mathbb{Q}}.$$

For the following Proposition we assume the hypothesis discussed in the introduction of [DLR1], and in particular, that $L(f \otimes g \otimes h, 1) = 0$. Recall the decomposition

$$P = P_{\beta\beta} \otimes e_{\beta\beta} + P_{\beta\alpha} \otimes e_{\beta\alpha} + P_{\alpha\beta} \otimes e_{\alpha\beta} + P_{\alpha\alpha} \otimes e_{\alpha\alpha},$$

and similarly for Q.

Define the regulators

$$\mathrm{Reg}_{g_\alpha}(V_{gh}) = \log_p(P_{\beta\beta})\log_p(Q_{\beta\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\beta\alpha})$$

and

$$\mathrm{Reg}_f(V_{gh}) = \log_p(P_{\beta\beta})\log_p(Q_{\alpha\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\alpha\alpha}).$$

To shorten notations, write

$$\log^{-+}(\kappa) = \langle \log_{\mathrm{BK}}(\kappa_p^g), \omega_f \otimes \eta_g \otimes \omega_h \rangle,$$

and whenever κ is crystalline, write $\log^{++}$ for the Bloch–Kato logarithm of [BSV2, Section 2.1], as recalled before during the proof of Proposition 4.3.

Proposition 5.2. *Assume that* $\mathrm{Reg}_{g_\alpha}(V_{gh})$, $\mathrm{Reg}_f(V_{gh}) \neq 0$. *Suppose that* two *of the following* three *equalities are true modulo* $L^\times$. *Then, the third automatically holds.*

(a)

$$\mathscr{L}_p{}^{g_\alpha}(f, g_\alpha, h_\alpha) = \frac{\log_p(P_{\beta\beta})\log_p(Q_{\beta\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\beta\alpha})}{\log_p(u_{g_\alpha})}.$$

(b)

$$\frac{\partial^2 \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, h_\alpha)}{\partial x^2}\Big|_{x=x_0} = \log_p(P_{\beta\beta})\log_p(Q_{\alpha\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\alpha\alpha}).$$

(c)

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}).$$

Proof. The proof is based on the study of the local cohomology class $\kappa_p(f, g_\alpha, h_\alpha)$ introduced in the preceding sections.

Observe that (a) and (b) are equivalent to

$$\log^{-+}(\kappa_p(f, g_\alpha, h_\alpha)) = \frac{\log_p(P_{\beta\beta})\log_p(Q_{\beta\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\beta\alpha})}{\log_p(u_{g_\alpha})} \pmod{L^\times}$$

and

$$\log^{++}(\kappa_p(f, g_\alpha, h_\alpha)) = \log_p(P_{\beta\beta})\log_p(Q_{\alpha\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\alpha\alpha}) \pmod{L^\times},$$

respectively, by virtue of the explicit reciprocity law (both in the usual version and *improved* version based on the techniques of Venerucci).

Let us define the local class

$$\kappa_0 = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot (\log_p(P_{\beta\beta}) \cdot Q - \log_p(Q_{\beta\beta}) \cdot P), \tag{20}$$

where we have implicitly identified a point over the elliptic curve with its image under the Kummer map; take then $\tilde{\kappa} = \kappa - \kappa_0$. The element $\tilde{\kappa}$ clearly belongs to the kernel of the Bloch–Kato logarithm $\log^{-+}$, that we have defined by

$$\log^{-+} : H^1(\mathbb{Q}_p, V_{fgh}) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha\beta}) \to \mathbb{C}_p,$$

the last map being the composition of the Perrin-Riou map and the pairing with the differentials $\omega_f \otimes \eta_{g_\alpha} \otimes \omega_{h_\alpha}$. Then, $\tilde{\kappa} = \lambda(\log_p(P_{\beta\alpha}) \cdot Q - \log_p(Q_{\beta\alpha}) \cdot P)$. But observe that by [BSV1, Corollary 8.2] we know that the cohomology class κ lies in the balanced part for the filtration attached to $H^1(\mathbb{Q}_p, V_{fgh})$ and hence $\tilde{\kappa}$ lies in the kernel of the map $\log^{--}$

$$\log^{--} : H^1(\mathbb{Q}_p, V_{fgh}) \xrightarrow{\mathrm{pr}^{--}} H^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha\alpha}) \to \mathbb{C}_p.$$

Hence, the non-vanishing of the regulator $\mathrm{Reg}_{g_\alpha}(V_{gh})$, implies that $\tilde{\kappa} = 0$ and therefore $\kappa = \kappa_0$.

From the same argument and under the assumption that $\mathrm{Reg}_f(V_{gh})$, the second equation yields

$$\kappa = \frac{1}{\Omega_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot (\log_p(P_{\beta\beta}) \cdot Q - \log_p(Q_{\beta\beta}) \cdot P) \pmod{L^\times}, \tag{21}$$

where again we have identified the points with their image under the Kummer map.

Now the statement is clear. For instance, if both (a) and (b) are true, comparing the previous expressions, we get

$$\log_p(u_{g_\alpha}) = \frac{\Xi_{g_\alpha}}{\Omega_{g_\alpha}} \pmod{L^\times}.$$

The proof of the other implications is equally straightforward. □

It would be interesting to prove an analogue result in a more general situation, beyond the case of split multiplicative reduction. The discussion around cohomology classes is still valid, but the point is that one needs a replacement for the results expressing the second derivative of $\mathscr{L}_p{}^f$ in terms of the Bloch–Kato logarithm of the cohomology class. While we can assure that the special value $\mathscr{L}_p{}^f$ is zero, it is not clear how to proceed with its derivatives.

Question 5.3. Is there a reciprocity law relating the second derivative of $\mathscr{L}_p{}^f$ (or some *variation* of it) with the logarithm $\log^{--}$ of the cohomology class $\kappa(f,g,h)$ in a *generic* situation (non exceptional zeros)?

5.2. **Case (a).** We assume first that $\alpha_g \alpha_h = 1$. The results we have proved until now showing a deep interaction between the value of the derivatives of $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$ and the value of $\mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})$ may be applied to study new instances of the elliptic Stark conjecture.

Let us analyze some particular cases describing the exact shape of the generalized cohomology classes. For example, according to the results of [BSV1], when g is a theta series attached to a quadratic field where the prime p is inert and $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$ with ψ_1 being a genus character, we have

$$\frac{d^2 \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, h_\alpha)}{dx^2}\Big|_{x=x_0} = \log^{++}(\kappa_p(f,g_\alpha,h_\alpha)) = \log(P_{\psi_1}^+)\cdot \log(P_{\psi_2}^+) \pmod{L^\times},$$

where $P_{\psi_i}^+ = P_{\psi_i} + \sigma_p P_{\psi_i}$, being $\sigma_p \in \mathrm{Gal}\,(H/\mathbb{Q})$ a Frobenius element at p.

Remark 5.4. This situation occurs in general when at least one of ψ_1 or ψ_2 is a genus character. See for example the discussion after [DLR1, Lemma 3.10] where the authors explain how the regulator of the elliptic Stark conjecture admits a particularly simple expression in this case.

However, from the results we already know around the elliptic Stark conjecture, one obtains that

$$\mathscr{L}_p{}^{g_\alpha}(f,g_\alpha,h_\alpha) = \frac{\log(P_{\psi_1}^+)\cdot\log(P_{\psi_2}^-)}{\mathcal{L}_{g_\alpha}} \pmod{L^\times}, \tag{22}$$

where with the previous notations, $P_{\psi_i}^- = P_{\psi_i} - \sigma_p P_{\psi_i}$. This is quite significant, since it establishes the elliptic Stark conjecture only up to a conjecture about periods of weight one modular forms.

Corollary 5.5. *Let g and h be theta series attached to a quadratic field (either real or imaginary) where the prime p remains inert, with $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$ and ψ_1 or ψ_2 being a genus character. Then, the elliptic Stark conjecture of* [DLR1] *is equivalent to the conjecture about periods of* [DR2].

Proof. This follows from the fact that part (b) of Proposition 5.2 holds in this setting. □

Moreover, we expect conjectural expressions for the generalized Kato classes. In particular, the previous result suggests the following conjecture.

Proposition 5.6. *In the setting of Proposition 5.2, if the formulas which appear in that statement are satisfied, then the equality*

$$\kappa(f,g_\alpha,h_{1/\alpha}) = \frac{1}{\Omega_{g_\alpha}\cdot\Omega_{h_\alpha}}\cdot(\log_p(P_{\beta\beta})\cdot Q - \log_p(Q_{\beta\beta})\cdot P),$$

holds in $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$ *up to multiplication by* $L^\times$.

Proof. This follows verbatim the proof of Proposition 5.2, using the third statement to simplify the different period relations. □

The same result holds for $\kappa(f, g_\beta, h_{1/\alpha})$.

5.3. **Case (b).** In the case where $\alpha_g \beta_h = 1$, the explicit reciprocity law gives a connection between $\mathscr{L}_p{}^{g_\alpha}$ and the Bloch–Kato logarithm of $\kappa(f, g_\alpha, h_{1/\beta})$, but unfortunately both the latter class and one of the Euler factors involved in the equality vanish. Therefore, that result is meaningless in this setting.

In previous sections we saw how to overcome that difficulty, proving a *derived reciprocity law* after having observed that certain Euler factors are analytic along the line $k+m = \ell+2$. There are two *natural* directions for considering the derivative over that plane (although of course it makes sense to take any combination of them): the line $(2, \ell, \ell)$ and the line $(k+1, k, 1)$; the former is not quite interesting since both the class $\kappa(f, g_\alpha, g^*_{1/\beta})$ and the Euler factor in the denominator of the Perrin-Riou map vanish identically. Hence, we may take derivative along $(k+1, k, 1)$ and we get an equality of the form

$$\mathcal{L} \cdot \mathscr{L}_p{}^{g_\alpha}(f, g_\alpha, h_{1/\beta}) = \log^{-+}(\kappa'_p(f, g_\alpha, h_{1/\beta})) \pmod{L^\times},$$

where $\mathcal{L}$ is the $\mathcal{L}$-invariant which already appeared in previous sections. Hence, if the elliptic Stark conjecture for $\mathscr{L}_p{}^{g_\alpha}$ were true, the class $\kappa'_p(f, g_\alpha, h_\alpha)$ could be expressed as a combination of points, normalized by appropriate $\mathcal{L}$-invariants. In particular, this would yield an equality of the form

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \frac{\mathcal{L}}{\Xi_{g_\alpha} \cdot \Omega_{h_{1/\beta}}} \cdot \frac{\log_p(P_{\beta\beta}) \cdot Q - \log_p(Q_{\beta\beta}) \cdot P}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{23}$$

One may obtain a symmetric expression for $\kappa'(f, g_\beta, h_{1/\alpha})$. Recall that this is the analogue of [RiRo1, Theorem B].

Conjecture 5.7. *The equality*

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \frac{\mathcal{L}}{\Xi_{g_\alpha} \cdot \Omega_{h_{1/\beta}}} \cdot \frac{\log_p(P_{\beta\beta}) \cdot Q - \log_p(Q_{\beta\beta}) \cdot P}{\log_p(u_{g_\alpha})} \pmod{L^\times}$$

holds in $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$.

As it was pointed out before, in the self dual case the product $\Xi_{g_\alpha}\Omega_{h_{1/\beta}}$ is an element of $L^\times$.

We finish our work with the following result.

Proposition 5.8. *Assume that Conjecture 5.7 is true. Then, the special value* $\mathscr{L}_p{}^{g_\alpha}$ *satisfies*

$$\mathscr{L}_p{}^{g_\alpha}(f, g_\alpha, h_\alpha) = \frac{\log_p(P_{\beta\beta})\log_p(Q_{\beta\alpha}) - \log_p(Q_{\beta\beta})\log_p(P_{\beta\alpha})}{\log_p(u_{g_\alpha})} \pmod{L^\times}.$$

Proof. This follows by applying the Bloch–Kato logarithm $\log^{-+}$ to the cohomology class $\kappa'(f, g_\alpha, h_{1/\beta})$, and using the derived reciprocity law of Theorem 3.10. □

The converse can also be established with some extra assumptions, including the conjecture about periods of [DR2].

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