Talk 9. Galois representations and Shimura varieties

§1. The case of modular curves

§2. Galois representations and automorphic forms (after Buzzard – Gee)

§3. Galois representations and Shimura varieties

§4. Some examples: the Hilbert and unitary cases

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§1. Modular curves: the Eichler–Shimura isomorphism

Toy case: $E/\mathbb{Q}$ elliptic curve.

Consider $E[\ell^n]$, where ℓ is a prime of good reduction.

Then $E[\ell^n] \simeq \mathbb{Z}/\ell^n \times \mathbb{Z}/\ell^n$ + action of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

$\hookrightarrow \rho_{E,n} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[\ell^n]) \simeq GL_2(\mathbb{Z}/\ell^n)$.

Taking inverse limits, we have $\rho_E : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_\ell)$.

"compatible system of Galois representations", i.e., for $p \neq \ell$, the characteristic polynomial of Frob_p has $\mathbb{Z}$-coefficients independent of ℓ.

Play the same game with $X_1(N)$, passing to $\mathrm{Jac}(X_1(N))$: gives $G_{\mathbb{Q}} \to GL_{2g}(\mathbb{Z}_\ell)$, "sum of 2-dimensional representations, one for each wt. 2 $\overset{\text{cusp}}{\text{mf}}$ f of level N".

↳ More generally, Eichler – Shimura theory (work with $\mathbb{C}$-coefficients)

- $V_k = \mathrm{Sym}^k \mathbb{C}^2$, (if $k=0$, this is just $\mathbb{C}$).
- $H^1(Y_\Gamma, V_k)$, where Y_Γ modular curve of level Γ. (Γ congruence subgroup).
- In terms of group cohomology as $H^1(Y_\Gamma, V_k) \cong H^1(\Gamma, V_k)$
  ↳ also via modular symbols, $\mathrm{Symb}_\Gamma(V_k) \cong H^1_c(Y_\Gamma, V_k)$

<u>Thm</u>: $\exists$ Hecke <u>equivariant</u> isomorphism

$$S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)} \oplus \mathrm{Eis}_{k+2}(\Gamma) \xrightarrow{\sim} H^1(Y_\Gamma, V_k)$$

other interesting cases: - Bianchi
- Hilbert
...

§2. Galois representations and automorphic forms

$$\begin{tikzcd}
& \text{Galois representations (and L-functions)} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Automorphic forms} \arrow[rr, leftrightarrow] & & \text{Shimura varieties}
\end{tikzcd}$$

<u>Problem</u>. Associate Galois representation to automorphic representations for an arbitrary connected reductive group over number fields.

- 1st option: work with <u>all</u> automorphic representations and associate to them representations of a (big) "Langlands group".
- 2nd option: restrict to "algebraic representations"
  ↳ work via fundamental group of some Tannakian category of motives
  ↳ work with l-adic representations.

Two different notions of algebraicity: <u>L-algebraic</u> and <u>C-algebraic</u> (essentially differ by a twist, see Buzzard - Gee). Neglect it along this talk.

<u>Important concept</u>: the L-group

- F number field, G connected reductive group/F, $G_F = \mathrm{Gal}(\overline{F}/F)$. Recall Xenia's talk.
- Root datum: maximal torus T in $G_{\overline{F}}$, $B \supset T$ Borel. A "based root datum" is $\Psi_0(G,B,T) := (X^*(T), \Delta^*(B), X_*(T), \Delta_*(B))$, the character/cocharacter group of T together w/ the roots/coroots simple and positive.
- $\Psi_0(G) := (X^*, \Delta^*, X_*, \Delta_*)$ projective limit via isomorphisms.
  $\hookrightarrow \mu_G : G_F \longrightarrow \mathrm{Aut}(\Psi_0(G))$.
- $\widehat{G}$ Langlands dual (root datum dual to that of G), $\Psi_0(\widehat{G}) = \Psi_0(G)^\vee$. $G_F \curvearrowright \widehat{G}$ via μ_G.

E.g. $GL(n)$, $SO(2n)$ are self-dual. $Sp(n)$ dual to $SO(2n+1)$

<u>Def'n.</u> ${}^LG$ pro-algebraic group $/\overline{\mathbb{Q}}$ given as

$${}^LG = \widehat{G} \rtimes G_F.$$

If G splits over F, ${}^LG = \widehat{G} \times G_F$.

<u>Def'n.</u> $f: G_F \to {}^LG$ L-homomorphism if composite w/ ${}^LG \to G_F$ is the identity.

v finite place of F st G is unramified

$\overline{F} \hookrightarrow \overline{F_v}$ (G_{F_v} subgroup of G_F)

<u>Def'n.</u> A Langlands-Satake parameter is a $\widehat{G}(\mathbb{C})$-conjugacy class of L-homomorphisms

$$s_v : W_{F_v} \longrightarrow {}^LG(\mathbb{C})$$

st the proj. to $\widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(F_v^{nr}/F_v)$ factors through W_{F_v}/I_{F_v} + certain semplicity condition.

<u>Rk.</u>: determined by $s_v(\mathrm{Frob}_v)$ in $\widehat{G}(\mathbb{C})$ if G splits$/F$.

The Buzzard – Gee conjecture

- $\Pi = \otimes' \Pi_v$, Satake parameters $(S_v)_{v \notin \Sigma}$ → bad places
- Harish – Chandra parameter for each ∞ place σ of F, Weyl group orbit $\lambda_\sigma \in X_\bullet(\hat{T}) \otimes \mathbb{C}$.

<u>Def'n</u> Π is L-algebraic if $\lambda_\sigma \in X_\bullet(\hat{T})$ $\forall \sigma$ infinite.

<u>Conj.</u> (BG) Suppose Π is L-algebraic automorphic rep'n. of $G(\mathbb{A}_F)$. Then, $\exists$ $E/\mathbb{Q}$ finite st the Satake parameters $v(\Pi_v)$ are all defined over E; and for any prime ℓ, and choice of embedding $\iota : E \hookrightarrow \overline{\mathbb{Q}}_\ell$, $\exists$ ~~admissible~~ L-homomorphism

$$r_\Pi : G_F \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$$

st $r_\Pi |_{G_{F_v}}$ is conjugate to $\iota(S_v)$ $\forall v \notin \Sigma$

§ 3. Galois representations and Shimura varieties

<u>Idea</u>: use Shimura datum to complete the diagram

$$\begin{tikzcd} r_\Pi : G_F \arrow[r] \arrow[dr] & {}^L G(\overline{\mathbb{Q}}_\ell) \arrow[d, "r_L"] \\ & \mathrm{Aut}(V) \end{tikzcd}$$

<u>Contruction of r_L</u>

(G, X), with X $G(\mathbb{R})$-conjugacy class of homomorphisms

$$h : \mathbb{S} \to G_\mathbb{R}$$

of real algebraic groups (remember, $\mathbb{S} = \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}} GL_{1/\mathbb{C}}$), Shimura datum.

From previous talks, $\mu_h(z) = h_\mathbb{C}(z, 1)$ cocharacter of $G_\mathbb{C}$

$$\begin{tikzcd} \mu_h : \mathbb{G}_m \arrow[r] & \mathbb{G}_m \times \mathbb{G}_m \arrow[r] & (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}} \arrow[d, "h"] \\ & & G_\mathbb{C} \end{tikzcd}$$

$$z \longmapsto (z, 1)$$

· Fix a pinning (T, B) of $G \rightsquigarrow (\hat{T}, \hat{B})$ of $\hat{G}$.
↳ notion of dominant weights / coweights, positive roots/coroots.
Let μ elt. in the conjugacy class of $(-\mu_h)_{h \in X}$ antidominant.

① μ gives antidominant integral weight $\hat{\mu}$ of $\hat{G}$ and hence unique irreducible representation

$$r : \hat{G} \to \mathrm{Aut}(V_\mu)$$

w/ $\hat{\mu}$ as extreme weight.

② extend to $r_L : {}^L G_F = \hat{G} \rtimes G_F \to \mathrm{Aut}(V_\mu)$
by letting G_F act trivially on $\hat{\mu}$.

Hence, a Shimura datum for G determines a Galois representation $G_F \to \mathrm{Aut}(V_\mu)$.

§4. Some examples: Hilbert and unitary cases

· Let K be a real quadratic field, and let F be a Hilbert modular eigenform of weight $(\underline{k}, \underline{t})$, with $k_1, k_2 \geq 2$.
· 2 different options for realising Galois representation

* B_1/K quaternion algebra split everywhere.
(The reflex field of the corresponding Shimura variety is $\mathbb{Q}$).
Carayol constructed ${}^L G(\mathbb{Q}_\ell) \to GL_2(\mathbb{Q}_\ell)$ associated Galois representation for F ($G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_p)$ w/ prescribed char. pol. for Frobenius).

* B_2/K quaternion algebra split at one prime, ramified at the other. Reflex field is now K)
Brylinski – Labesse, Nekovar constructed the Asai representation ${}^L G(\mathbb{Q}_\ell) \to GL_4(\mathbb{Q}_\ell)$.

So in this case [redacted] different Galois representations, coming from choice of different Shimura data.

Similar for unitary groups. $U(p,q)$ vs $U(p',q')$ w/ $p<q$, $p'<q'$, $p+q=p'+q'$. Now K quad. imag.
Given a Hecke eigensystem[1], we expect to realise it in ${}^LG(\mathbb{Q}_\ell)$. [redacted]

<u>But</u> the highest weight representation is different

For $U(1,4)$ get $G_{\mathbb{Q}} \to GL(\mathbb{Q}_\ell^5) = GL_5(\mathbb{Q}_\ell)$, while
for $U(2,3)$ get $G_{\mathbb{Q}} \to GL(\mathrm{Sym}^2 \mathbb{Q}_\ell^5)$ sym. square rep'n.

Here even the reflex fields $\underbrace{\text{agree}[2], \text{but}}_{\text{(K itself)}}$ the mult. of the <u>highest weight representation</u> is different.

↳ for $U(p,q)$ we can represent it as the $(p+q)$-tuple
$$(\underbrace{1,\dots,1}_{p},\underbrace{0,\dots,0}_{q})$$

* It is the existence of canonical models discussed in Elvira and Chris' talk what allows us to attach Galois representations to the étale cohomology of Shimura varieties over their reflex fields.

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[1] This requires some mild assumptions on the automorphic representation. In general, if π is discrete series it transfers to any unitary group of arbitrary signature.

[2] Provided that $p \neq q$.