ALGEBRAICITY OF L-VALUES FOR $\mathrm{GSp}_4 \times \mathrm{GL}_2$ AND $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$

DAVID LOEFFLER AND ÓSCAR RIVERO

Abstract. We prove algebraicity results for critical L-values attached to the group $\mathrm{GSp}_4 \times \mathrm{GL}_2$, and for Gan–Gross–Prasad periods which are conjecturally related to central L-values for $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. A key aspect is the computation of certain archimedean zeta integrals, whose p-adic counterparts are also studied in this note. This is crucially used in a sequel to this paper in order to construct a new p-adic L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

Contents

1. Introduction

1.1. **Critical L-values for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.** Let $\pi \times \sigma$ be an automorphic representation of $\mathrm{GSp}_4 \times \mathrm{GL}_2$. We can attach to π and σ a degree 8 L-function $L(\pi \times \sigma, s)$, associated to the tensor product of the natural degree 4 (spin) and degree 2 (standard) representations of the L-groups of GSp_4 and GL_2.

If π and σ are algebraic, then this L-function is expected to correspond to a motive, and in particular we can ask whether it has critical values. More precisely, suppose that (the L-packet of) π corresponds to a holomorphic Siegel modular form of weight (k_1, k_2), with $k_1 \geqslant k_2 \geqslant 2$; and that σ corresponds to a modular form of weight $\ell \geqslant 1$. Then we expect there to exist motives $M(\pi)$ (of motivic weight $k_1 + k_2 - 3$) and $M(\sigma)$ (of motivic weight $\ell - 1$) such that

$$L(\pi \times \sigma, s) = L\left(M(\pi) \otimes M(\sigma), s + \tfrac{w}{2}\right), \qquad w = k_1 + k_2 + \ell - 4.$$

For $L(\pi \times \sigma, s)$ to be a critical value, we must have $s = -\frac{w}{2} \bmod \mathbf{Z}$, and s must satisfy various inequalities depending on how the weights of π and σ interlace. For fixed (k_1, k_2), the allowed pairs (s, ℓ) form three disjoint polygonal regions in the plane (all symmetric about the line $s = \frac{1}{2}$); for compatibility with the conjectures of [LZ20b] on Gross–Prasad periods (see below), we denote these by (A), (D) and (F). The inequalities defining these regions are given in Table 1. The main goal of this paper is to prove an algebraicity result for weights in region (D), extending earlier partial results due to Böcherer–Heim and Saha which we recall in more detail below. This algebraicity result will be used in a sequel paper to study p-adic interpolation of critical L-values in region (D).

2010 *Mathematics Subject Classification.* 11F46; 11F70.

Key words and phrases. algebraicity of L-values, Siegel Shimura varieties, zeta integrals.

Supported by ERC Consolidator grant "Shimura varieties and the BSD conjecture" (D.L.) and Royal Society Newton International Fellowship NIF\R1\202208 (O.R.). This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2023 semester.

Table 1. Critical regions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$

Region	Range of ℓ	Range of s
(A)	$k_1 + k_2 - 1 \leqslant \ell$	$\lvert 2s-1\rvert \leqslant \ell - (k_1 + k_2 - 1)$
(D)	$k_1 - k_2 + 3 \leqslant \ell \leqslant k_1 + k_2 - 3$	$\lvert 2s-1\rvert \leqslant \min\left(k_1 + k_2 - 3 - \ell, \ell - (k_1 - k_2 + 3)\right)$
(F)	$1 \leqslant \ell \leqslant k_1 - k_2 + 1$	$\lvert 2s-1\rvert \leqslant k_1 - k_2 + 1 - \ell$

(In the "missing" cases $\ell = k_1 \pm (k_2 - 2)$, the L-function has no critical values.)

1.2. **Gross–Prasad periods for** $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$**.** The second problem we consider here is to study the *Gross–Prasad period* attached to a cuspidal automorphic representation π of GSp_4 and two cuspidal automorphic representations σ_1, σ_2 of GL_2, which is defined by

$$\mathcal{P}(\varphi, \psi_1, \psi_2) = \int_{[H]} \varphi(h)\psi_1(h_1)\psi_2(h_2)\mathrm{d}h,$$

for forms $\varphi \in \pi$ and $\psi_i \in \sigma_i$. Here H denotes the group $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, and $[H]$ the adelic symmetric space $\mathbf{R}^\times H(\mathbf{Q})\backslash H(\mathbf{A})$. (We may assume $\chi_\pi \chi_{\sigma_1}\chi_{\sigma_2} = 1$, since the integral is trivially zero otherwise.) The Gross–Prasad period vanishes unless the central characters satisfy the condition $\varepsilon(\pi_v \times \sigma_{1,v} \times \sigma_{2,v}) = +1$ for all places v. If this condition holds, the Gross–Prasad conjecture for $\mathrm{SO}_4 \times \mathrm{SO}_5$ predicts that $\mathcal{P}(-)$ is non-zero if and only if $L(\pi \times \sigma_1 \times \sigma_2, \frac{1}{2}) \neq 0$, and Ichino and Ikeda [II10] have formulated a precise conjecture relating $|\mathcal{P}(\varphi, \psi_1, \psi_2)|^2$ to the central L-value. (More precisely, the original works of Gross–Prasad and Ichino–Ikeda apply when $\chi_\pi = 1$, so π factors through SO_5 and $\sigma = \sigma_1 \boxtimes \sigma_2$ through its subgroup SO_4; the more general formulation we use here is due to Emory [Emo19].) We shall prove here results for the periods $\mathcal{P}(-)$, which are not logically dependent on the conjectured relation to L-values, but nonetheless the Gross–Prasad and Ichino–Ikeda conjectures are the key motivation for studying these periods.

If we take π to have weight (k_1, k_2), as above, and σ_1, σ_2 to correspond to holomorphic cuspforms of weights c_1, c_2, then we have 9 different cases $\{a, a', b, b', c, d, d', e, f\}$ depending on the inequalities satisfied by (c_1, c_2) and (k_1, k_2). See Figure 2 of [LZ20b] for a diagram illustrating these. In cases $\{b, b', e\}$ the Gross–Prasad period is automatically zero. In the remaining six cases one expects a rationality result for the Gross–Prasad period, whose formulation will depend on which case we consider; this is implicit in Conjecture 4.1.2 of [LZ20b].

The relation between these periods and the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ problem above is via Novodvorsky's integral formula for the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-function [Nov79]. Novodvorsky's integral can be seen as a "degenerate case" of the Gross–Prasad period, in which the cuspidal GL_2-representation on the first factor of H is replaced by a space of Eisenstein series. This relation allows the two rationality problems to be treated in parallel. If s is a critical value for the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-function, then the Eisenstein automorphic representation playing the role of σ_1 has weight $1 + |2s-1|$; and the cases (A), (D), (F) of the previous section correspond to requiring that the pair $(c_1, c_2) = (1 + |2s-1|, \ell)$ should satisfy the inequalities $(a), (d), (f)$ respectively.

1.3. **Algebraicity results (Eisenstein case).** Let π, σ be as in Section 1.1. We assume that $\ell \neq k_1 \pm (k_2 - 2)$, so that critical values exist and we are in one of the cases (A), (D), (F). Let m denote the sum of the four smallest Hodge numbers of $M(\pi) \otimes M(\sigma)$, so that

$$m = \begin{cases} 2k_1 + 2k_2 - 6 & \text{case } (A), \\ k_1 + k_2 + \ell - 4 & \text{case } (D), \\ 2k_2 + 2\ell - 6 & \text{case } (F). \end{cases}$$

Conjecture 1.1. *(1) There exists a period $\Omega(\pi \times \sigma) \in \mathbf{C}^\times$ with the following property: for every $j \in \mathbf{Z}$ such that $s = -\frac{w}{2} + j$ is a critical value, we have*

$$\frac{L(\pi \times \sigma, -\frac{w}{2} + j)}{(-2\pi i)^{4j-m}\Omega(\pi \times \sigma)} \in \overline{\mathbf{Q}}.$$

(2) For χ a Dirichlet character we have $\Omega(\pi \times \sigma \times \chi) = \Omega(\pi \times \sigma)$ mod $\overline{\mathbf{Q}}^\times$.

Of course, one would also like to have a tolerably explicit form for the period $\Omega(\pi\times\sigma)$. In the special case when $k_1 = k_2 = \ell$ and π and σ are unramified at all finite places, Yoshida [Yos01] has shown that Deligne's general algebraicity conjectures [Del79] imply an explicit form for the periods $\Omega(\pi\times\sigma)$ in terms of Petersson norms of holomorphic eigenforms. Yoshida's hypotheses rule out case (F); the remaining cases (A) and (D) correspond, respectively, to cases (A) and (B) of [Yos01, Theorem 13].

We may suppose without loss of generality that π is *tempered*, since Arthur's classification of the discrete spectrum [Art04, GT18] shows that for non-tempered π, the L-function can be expressed as a product of automorphic L-functions for GL_2 and $\mathrm{GL}_2\times\mathrm{GL}_2$, and the algebraicity properties of these L-values are well-understood.

1.3.1. *Known results: case* (F). For case F, the conjecture is known in full: it is proved in [LPSZ19], building on Harris' study of "occult periods" for the degree 4 L-function of GSp_4 in [Har04]. In this case, the period $\Omega(\pi\times\sigma)$ depends only on π (not on σ), and is defined using the $\overline{\mathbf{Q}}$-structure on H^2 of coherent automorphic sheaves on a toroidal compactification of the Siegel modular threefold.

1.3.2. *Known results: case* (D). As far as we have been able to determine, the strongest algebraicity results for case D in the literature to date are the following:

- Böcherer–Heim [BH06] consider the case $k_1 = k_2 = 2k_G$ and $\ell = 2k_h$ for some integers k_G, k_h, and π and σ are unramified at all finite primes, generated by holomorphic eigenforms G for $\mathrm{Sp}_4(\mathbf{Z})$ and h for $\mathrm{SL}_2(\mathbf{Z})$ respectively. For such π, σ, they prove part (1) of Conjecture 1.1 for the explicit period
$$\Omega(\pi\times\sigma)^{\mathrm{BH}} := \langle G, G\rangle\cdot\langle h, h\rangle,$$
where $\langle -,-\rangle$ denotes the Petersson scalar product (and G and h are normalised to have their Fourier coefficients in $\overline{\mathbf{Q}}$). Their result assumes that the first Fourier–Jacobi coefficient of G is non-vanishing.
- Saha [Sah10] proves an analogous result with slightly stronger assumptions on the weight (he assumes that $k_1 = k_2 = \ell \geqslant 6$), but less restrictive conditions on the level (allowing π and σ to be either the Steinberg representation, or its unramified quadratic twist, at some finite places). This refines earlier works of Furusawa, Pitale–Schmidt, and Saha himself. As in Böcherer–Heim, Saha assumes a non-vanishing hypothesis for a certain Fourier coefficient (depending on the levels of π and σ); and the period he uses is again $\langle G, G\rangle\cdot\langle h, h\rangle$.

Note that both of these works assume in particular that $k_1 = k_2$, so π corresponds to a scalar-valued holomorphic Siegel eigenform (rather than vector-valued).

1.3.3. *The mystery of case* (A). For case (A), Yoshida's work predicts that the period $\Omega(\pi\times\sigma)$ should be independent of π and is given by $\langle g, g\rangle^2$, where g is the normalised newform generating σ as before. Böcherer–Heim verify the conjecture in this range assuming π is a Saito–Kurokawa lift, but their method does not work for general-type π. We also have no new results in this case, and we mention it here in the hope that future authors may be able to shed light upon it.

1.4. **Our results.** In this paper, we focus on case (D), or more precisely on the following sub-case:

Definition 1.2. *Say we are in case (D^-) if (k_1, k_2, ℓ, s) satisfy the inequalities of case (D) together with the additional condition $\ell \leqslant k_1$.*

For weights in this region we prove the following:

Theorem 1.3. *Assume the weights of (π,σ) are in case (D^-), and the following local hypothesis holds:*

- *For each prime ℓ (if any) such that σ_ℓ is supercuspidal, the central character of π_ℓ is a square in the group of characters of $\mathbf{Q}_\ell^\times$.*

Then Conjecture 1.1 holds for $\pi\times\sigma$, with a period of the form $\Omega(\pi\times\sigma) = \Omega^{(1)}(\pi)\cdot\langle h, h\rangle$, where $\langle h, h\rangle$ denotes the Petersson norm of the normalised newform generating σ, and $\Omega^{(1)}(\pi)$ is a period depending only on π (defined using H^1 of coherent sheaves on the compactified Siegel threefold).

The local hypothesis is required in order to show that Novodvorsky's zeta integral computes the correct L-factors at the bad primes. It is, of course, vacuous if π has trivial central character. We hope that future work will allow this assumption to be removed.

Our methods are very different from those of [BH06] or [Sah10]. Where those works use integral formulae involving holomorphic (or nearly-holomorphic) cusp forms and Eisenstein series, we instead work with a non-holomorphic cusp form for GSp_4 belonging to the unique *globally generic* representation in the same L-packet as π. In particular, our results are unconditional, not depending on any non-vanishing assumptions on Fourier coefficients[1] as needed in the prior works cited.

Remark 1.4. Comparing our results with those of [BH06, Sah10] in the case $k_1 = k_2$, and supposing that the critical values do not all vanish (which is automatic if $\ell > k_1 - k_2 + 3$), we can deduce that the coherent H^1 period $\Omega^{(1)}(\pi)$ is a $\overline{\mathbf{Q}}^\times$-multiple of $\langle G, G\rangle$, for an appropriately normalised Siegel eigenform G generating π. This period relation is not at all obvious a priori. It would be interesting to extend this result to vector-valued forms (and to automorphic representations of higher levels which are not covered by the results of Böcherer–Heim and Saha). ◇

If $k_2 > 3$, then the set of (s, ℓ) satisfying (D^-) is a strict subset of the (s, ℓ) satisfying (D); and the methods used in this article do not apply for the remaining cases. It seems likely that these remaining cases will require a theory of Maass–Shimura-type differential operators acting on H^1 of Siegel threefolds; this is beyond the scope of the present work.

1.5. **Connection with other works.** In a sequel to this paper [LR23], we build on the algebraicity results developed here in order to define a p-adic L-function interpolating L-values along the "edge" of region (D) (that is, with $s = \frac{k_1+k_2-2-\ell}{2}$).

This computation requires the evaluation of a certain local zeta integral at the p-adic place (which gives the Euler factor $\mathcal{E}^{(D)}(\pi \times \sigma)$ appearing in the interpolation property of our p-adic L-function). We carry out this local computation here, rather than in the sequel paper, since its proof has much in common with the local Archimedean computation needed to prove Theorem 1.3 (and little in common with the p-adic interpolation computations which form the bulk of the sequel paper).

Acknowledgements. The authors would like to thank Sarah Zerbes for informative conversations related to this work.

2. The groups G and H and their Shimura varieties

2.1. **Groups and parabolics.** We denote by G the group scheme GSp_4 (over $\mathbf{Z}$), defined with respect to the anti-diagonal matrix $J = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}$; and we let ν be the multiplier map $G \to \mathbf{G}_m$. We define $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, which we embed into G via the embedding

$$\iota : \left[\begin{pmatrix} a & b\\ c & d\end{pmatrix}, \begin{pmatrix} a' & b'\\ c' & d'\end{pmatrix}\right] \mapsto \begin{pmatrix} a &&& b\\ & a' & b' &\\ & c' & d' & \\ c &&& d\end{pmatrix}.$$

We sometimes write H_i for the i-th GL_2 factor of H. We write T for the diagonal torus of G, which is contained in H and is a maximal torus in either H or G.

We write B_G for the upper-triangular Borel subgroup of G, and P_{Si} and P_{Kl} for the standard Siegel and Klingen parabolics containing B, so

$$P_{\mathrm{Si}} = \begin{pmatrix} \star&\star&\star&\star\\ \star&\star&\star&\star\\ &&\star&\star\\ &&\star&\star \end{pmatrix}, \qquad P_{\mathrm{Kl}} = \begin{pmatrix} \star&\star&\star&\star\\ &\star&\star&\star\\ &\star&\star&\star\\ &&&\star \end{pmatrix}.$$

We write $B_H = \iota^{-1}(B_G) = \iota^{-1}(P_{\mathrm{Si}})$ for the upper-triangular Borel of H.

[1] A conjecture of Böcherer relates this non-vanishing property of Fourier coefficients to the non-vanishing of central L-values of suitable twists of π. This conjecture of Böcherer has been proved in certain cases by Furusawa–Morimoto, but only under rather strong assumptions on the level and weight of π. Moreover, the vanishing property of twisted central L-values is also far from being obvious.

In this paper P_{Si} will be much more important than P_{Kl} (in contrast to [LPSZ19]). We have a Levi decomposition $P_{\mathrm{Si}} = M_{\mathrm{Si}} N_{\mathrm{Si}}$, with $M_{\mathrm{Si}} \cong \mathrm{GL}_2 \times \mathrm{GL}_1$, identified as a subgroup of G via

$$(A, u) \mapsto \begin{pmatrix} A & \\ & uA' \end{pmatrix}, \qquad A' \coloneqq \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \cdot {}^t A^{-1} \cdot \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

The intersection $B_M \coloneqq M \cap B_G$ is the standard Borel of M; its Levi factor is T.

2.2. **Flag varieties and Bruhat cells.** We write FL_G for the Siegel flag variety $P\backslash G$, with its natural right G-action. There are four orbits for the Borel B_G acting on FL_G, the *Bruhat cells*, represented by a subset of the Weyl group of G, the *Kostant representatives*, which are the smallest-length representatives of the quotient $W_M \backslash W_G$. We denote these by $w_0, \dots, w_3$; see [LZ21] for explicit matrices. Note that the cell $C_{w_i} = P\backslash P w_i B_G \subset \mathrm{FL}_G$ has dimension $\ell(w_i) = i$.

Analogously, for the H-flag variety $\mathrm{FL}_H = B_H\backslash H$, we have 4 Kostant representatives $w_{00} = \mathrm{id}$, $w_{10} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{id}\right)$, similarly w_{01}, w_{11} (with the cell $C_{w_{ij}}$ having dimension $i + j$). (This is the whole of the Weyl group of H, since the Levi subgroup of $M_H = T$ is trivial.)

2.3. **Representations.** We retain the conventions about algebraic weights and roots of [LZ21]. In particular, we identify characters of T with triples of integers $(r_1, r_2; c)$, with $r_1 + r_2 = c$ modulo 2 corresponding to $\mathrm{diag}(st_1, st_2, st_2^{-1}, st_1^{-1}) \mapsto t_1^{r_1} t_2^{r_2} s^c$. With our present choices of Borel subgroups, a weight $(r_1, r_2; c)$ is dominant for H if $r_1, r_2 \geqslant 0$, dominant for M_G if $r_1 \geqslant r_2$, and dominant for G if both of these conditions hold. (We frequently omit the central character c if it is not important in the context.)

2.4. **Models of Shimura varieties.** Let K be a neat open compact subgroup of $\mathrm{GSp}_4(\mathbf{A}_{\mathrm{f}})$. Denote by $Y_{G,\mathbf{Q}}$ the canonical model over $\mathbf{Q}$ of the level K Shimura variety. It is a smooth quasiprojective threefold, whose complex points are canonicallly identified with a double-coset space of $G(\mathbf{A})$, as discussed in [LPSZ19, §2.3]. We write $Y_{H,\mathbf{Q}}$ for the canonical $\mathbf{Q}$-model of the Shimura variety for H of level $K_H = K \cap H(\mathbf{A}_{\mathrm{f}})$, which is a moduli space for ordered pairs of elliptic curves with level structure. Further, there is a morphism of algebraic varieties $\iota : Y_{H,\mathbf{Q}} \to Y_{G,\mathbf{Q}}$.

After choosing a suitable combinatorial datum (a rational polyhedral cone decomposition), we can define a smooth compactification $X_{G,\mathbf{Q}}$ of $Y_{G,\mathbf{Q}}$. This depends on the choice of cone decomposition, but we shall not indicate this in the notation, since the choice of cone decomposition will remain fixed throughout. As usual, we denote by D the boundary divisor.

The cone decomposition for G naturally determines a cone decomposition for H and hence a compactification $X_{H,\mathbf{Q}}$ of $Y_{H,\mathbf{Q}}$, and the embedding ι extends to a finite morphism $X_{H,\mathbf{Q}} \to X_{G,\mathbf{Q}}$ (which we also denote by ι). One can in fact always choose the cone decomposition in such a way that this map of toroidal compactifications is a closed immersion [Lan19], although we do not need this here.

Remark 2.1. If K_H is the product of subgroups $K_{H,1} \times K_{H,2}$, then $Y_{H,\mathbf{Q}}$ is a product of two modular curves. Each of these has a canonical compactification given by adjoining finitely many cusps; but $X_{H,\mathbf{Q}}$ may not be the product of these compactified modular curves (depending on the choice of the toroidal boundary data). In general $X_{H,\mathbf{Q}}$ will be obtained from the product of compactified modular curves by a finite sequence of blow-ups concentrated above points of the form (cusp) $\times$ (cusp). ◇

2.5. **Coefficient sheaves.** We adopt the conventions recalled in [LZ21, §2.5.1]. For our further use, recall that the Weyl group acts on the group of characters $X^*(T)$ via $(w \cdot \lambda)(t) = \lambda(w^{-1} t w)$. As discussed in loc. cit., we can define explicitly $w_G^{\max}$, the longest element of the Weyl group, as well as $\rho = (2, 1; 0)$, which is half the sum of the positive roots for G.

There is a functor from representations of P_G to vector bundles on $X_{G,\mathbf{Q}}$; and we let $\mathcal{V}_\kappa$, for $\kappa \in X^\bullet(T)$ that is M_G-dominant, be the image of the irreducible M_G-representation of highest weight κ. Given an integral weight $\nu \in X^\bullet(T)$ such that $\nu + \rho$ is dominant, we define

$$\kappa_i(\nu) = w_i(\nu + \rho) - \rho, \quad 0 \leqslant i \leqslant 3,$$

where as usual ρ is half the sum of the positive roots. These are the weights κ such that representations of infinitesimal character $\nu^\vee + \rho$ contribute to $R\Gamma(S_K^{G,\mathrm{tor}}, \mathcal{V}_\kappa)$; if ν is dominant (i.e. $r_1 \geqslant r_2 \geqslant 0$), they are the weights which appear in the *dual BGG complex* computing de Rham cohomology with coefficients in the algebraic G-representation of highest weight ν. See [LZ21, §2.5.2] for explicit formulae.

2.6. **Cohomology.** According to the previous discussion, if V is an algebraic representation of P_S over $\mathbf{Z}_{(p)}$, we have a vector bundle $\mathcal{V}$ on X_G defined by $\mathcal{V} := V \times^{P_S} \mathcal{T}_G$, where $\mathcal{T}_G$ is the canonical P_S-torsor over X_G.

The Zariski cohomology groups $H^i(X_G, \mathcal{V})$ and $H^i(X_G, \mathcal{V}(-D))$ are independent, up to canonical isomorphism, of the choice of cone decomposition Σ for the compactification, and have actions of prime-to-p Hecke operators $[KgK]$, for $g \in G(\mathbf{A}_{\mathrm{f}}^p)$. The same is true for H in place of G, and hence there are morphisms of sheaves

$$H^i(X_G, \mathcal{V}) \xrightarrow{\iota^*} H^i(X_H, \mathcal{V}_{B_H}) \tag{1}$$

and also

$$H^i(X_H, \mathcal{V}_{B_H} \otimes \alpha_{G/H}^{-1}) \xrightarrow{\iota_*} H^{i+1}(X_G, \mathcal{V}) \tag{2}$$

for $0 \leqslant i \leqslant 2$ and any P_S-representation V. Here, $\alpha_{G/H}$ denotes the character $(1,1;0)$ of B_H, the Borel subgroup of H. These maps will play a key role later for defining the appropriate pairings involved in the main constructions of the note.

Let π (resp. σ) be a cuspidal automorphic representation of G (resp. GL_2), and consider the arithmetic normalisation of the finite part, $\pi'_{\mathrm{f}} := \pi_{\mathrm{f}} \otimes \|\cdot\|^{-(r_1+r_2)/2}$ (resp. $\sigma'_{\mathrm{f}} := \sigma_{\mathrm{f}} \otimes \|\cdot\|^{-c_1/2}$. Let L_1 stand for the irreducible M_S-representation with highest weight $L_1 : \lambda(r_1+3, 1-r_2)$. Similarly, L_2 is the irreducible M_S-representation with highest weight $L_2 : \lambda(r_2+2, -r_1)$. The following result is a consequence of Arthur's classification.

Theorem 2.2. *Let $i = 1, 2$. If π is of general type or of Yoshida type, then π'_{f} appears with multiplicity one as a Jordan–Hölder factor of the $G(\mathbf{A}_{\mathrm{f}})$-representations*

$$H^{3-i}(X_{G,\mathbf{Q}}, \mathcal{V}_{\kappa_i(\nu)}(-D)) \otimes \mathbf{C} \quad \text{and} \quad H^{3-i}(X_{G,\mathbf{Q}}, \mathcal{V}_{\kappa_i(\nu)}) \otimes \mathbf{C}.$$

Moreover, it appears as a direct summand of both representations, and the map between the two is an isomorphism on this summand. If L is any irreducible representation of M_S which is not isomorphic to $V_{\kappa_i(\nu)}$, then the localisations of $H^{3-i}(X_{G,\mathbf{Q}}, [L](-D))$ and $H^{3-i}(X_{G,\mathbf{Q}}, [L])$ at the maximal ideal of the spherical Hecke algebra associated to the L-packet of π are zero for all i.

We write $H^i(\pi_{\mathrm{f}})$ for the π'_{f}-isotypical component of $H^i(X_{G,E}, [L_1](-D))$, for some number field E over which π'_{f} is definable.

3. A pairing on coherent cohomology

In this section we define a pairing between coherent cohomology groups which we will later use to study L-values for region (D^-).

3.1. **Automorphic forms as coherent cohomology classes.** We fix a weight $\nu = (r_1, r_2; c) = (k_1 - 3, k_2 - 3; c)$, for integers $k_1 \geqslant k_2 \geqslant 2$. Then there are two discrete-series representations of $\mathrm{GSp}_4(\mathbf{R})$ of infinitesimal character $\nu^\vee + \rho$: a holomorphic discrete series π_∞^H, corresponding classically to holomorphic Siegel modular forms of weight (k_1, k_2), which contributes to cohomology in degrees 0 and 3; and a generic discrete series π_∞^W, which contributes in degrees 1 and 2.

More canonically, we can write this as follows. Let $K_\infty = \mathbf{R}^\times \cdot U_2(\mathbf{R})$ denote the maximal compact-mod-centre subgroup of $G(\mathbf{R})_+$. The representation π_∞^W has two direct summands as a $G(\mathbf{R})_+$-representation, $\pi_\infty^W = \pi_{\infty,1} \oplus \pi_{\infty,2}$, which have minimal K_∞-types $\tau_1 = (r_1+3, -r_2-1)$ and $\tau_2 = (r_2+1, -r_1-3)$, respectively. Since the minimal K_∞-type in an irreducible discrete series has multiplicity 1, we have $\dim \mathrm{Hom}_{K_\infty}(\tau_i, \pi_\infty) = 1$ for $i = 1, 2$. Then, for each automorphic representation π whose Archimedean component is π_∞^W, we have a canonical isomorphism of irreducible smooth $G(\mathbf{A}_{\mathrm{f}})$-representations

$$\mathrm{Hom}_{K_\infty}(\tau_i, \pi)\{\frac{r_1+r_2}{2}\} \cong H^{3-i}(X_{G,\mathbf{Q}}, \mathcal{V}_{\kappa_i}(-D))_{\mathbf{C}}[\pi_{\mathrm{f}}].$$

Remark 3.1. For weights (k_1, k_2) sufficiently far from the walls of the Weyl chamber (so there are no non-tempered representations contributing to the cohomology), this is proved in [HK92]. It follows from the results of [Su19], together with Arthur's classification of discrete-series representations of GSp_4, that the result in fact applies for all weights. ◇

Given $\xi \in H^{3-i}(X_{G,\mathbf{Q}}, \mathcal{V}_{\kappa_i}(-D))_{\mathbf{C}}[\pi_{\mathrm{f}}]$, we denote by F_ξ the corresponding homomorphism $\tau_i \to \pi$; we may consider F_ξ as a harmonic vector-valued cusp form on G, taking values in the representation $\tau_i^\vee$.

We use similar notations for GL_2: for any $k \geqslant 1$, the space $H^0(X_{\mathrm{GL}_2}, \mathcal{V}_{(-k)})$ is isomorphic to weight k modular forms; and $H^1(X_{\mathrm{GL}_2}, \mathcal{V}_{(k-2)})$ is isomorphic to the space of anti-holomorphic modular forms of weight $-k$ (i.e. complex conjugates of holomorphic forms of weight k). We write $\omega \to F_\omega$ and $\eta \to F_\eta$ for these isomorphisms.

3.2. **Whittaker periods.** Suppose π is a cuspidal automorphic representation with Archimedean component π_∞^W, and which is globally generic. Let E be the coefficient field of π.

If we choose $j \in \{1, 2\}$ then there is a canonical basis of $\mathrm{Hom}_{K_\infty}(\tau_{3-j}, \mathcal{W}(\pi_\infty^W))$ computed by Moriyama (which we shall recall in more detail below). From this we obtain two E-rational structures on $\mathrm{Hom}_{K_\infty}(\tau_j, \pi)$: one via the isomorphism to coherent H^j, and one via tensoring Moriyama's basis vector at ∞ with the canonical E-structure on the Whittaker model of π_{f}. Since π_{f} is irreducible, these must differ by a constant in $\mathbf{C}^\times/E^\times$.

Notation. *We denote this scalar factor by* $\Omega_\pi^{(j)} \in \mathbf{C}^\times/E^\times$*, the* H^j Whittaker period *of* π*.*

The period denoted by Ω_π^W in [LPSZ19] (appearing in rationality results for case (F)) is the H^2 Whittaker period. In the present work, it is the H^1 Whittaker period which appears instead. (It is far from clear *a priori* how these periods are related to each other.)

Similar considerations apply to holomorphic cuspidal GL_2 representations σ. In this case we obtain a period for H^0 and a period for H^1. We may choose our standard Whittaker functions at ∞ so that the Whittaker-rational classes are those whose q-expansions have coefficients in E. By the q-expansion principle the H^0 Whittaker period is just 1. On the other hand, since the Serre duality pairing on coherent cohomology preserves the E-rational structures, and this duality pairing corresponds to the Petersson product on automorphic forms, the H^1 Whittaker period must be given by the Petersson norm of the normalised newform generating σ.

3.3. **Pullback to H.** Now let (c_1, c_2) be a pair of integers, with $c_1 + c_2 = k_1 + k_2 \bmod 2$, and satisfying the inequalities

$$1 \leqslant c_1, \quad k_1 - k_2 + 2 \leqslant c_2 - c_1, \quad c_2 \leqslant k_1$$

defining the region (D^-). We denote by λ the weight $(-c_1, c_2 - 2)$ for H. We want to define a pairing

$$H^1\left(X_{G,\mathbf{Q}}, \mathcal{V}^G_{\kappa_2}(-D)\right) \times H^1\left(X_{H,\mathbf{Q}}, \mathcal{V}^H_\lambda\right) \to \mathbf{Q}. \tag{3}$$

Let us define $t = \frac{(c_2-c_1)-(k_1-k_2+2)}{2} \geqslant 0$.

3.4. **The $t = 0$ case.** Let us first suppose that we have $c_2 - c_1 = k_1 - k_2 + 2$; this is the situation studied in §2.5 of [HK92]. We have $\kappa_2(\nu) = (k_2 - 4, -k_1)$ and one computes that the pullback of $\mathcal{V}_{\kappa_2(\nu)}$ to H is given by the direct sum

$$\bigoplus_{0 \leqslant j \leqslant k_1+k_2-4} \mathcal{V}^H_{(k_2-4-j, j-k_1)}.$$

Since $c_2 \leqslant k_1$, this implies that one of the direct summands in $\iota^*(\mathcal{V}_{\kappa_2(\nu)})$ is $\mathcal{V}^H_{(c_1-2, -c_2)}$; so we obtain a pullback map

$$\begin{aligned}\iota^* : H^1(X_{G,\mathbf{Q}}, \mathcal{V}^G_{\kappa_2(\nu)}(-D)) &\to H^1(X_{H,\mathbf{Q}}, \mathcal{V}^H_{(c_1-2,-c_2)}(-D)) \\ &= \left[H^1(X_{H,\mathbf{Q}}, \mathcal{V}^H_{(-c_1,c_2-2)})\right]^\vee \quad \text{(by Serre duality)},\end{aligned}$$

defining a pairing between the groups (3), which can be explicitly expressed as the integral

$$\langle \xi, \omega \boxtimes \eta \rangle = \frac{1}{(2\pi i)^2} \int_{\mathbf{R}^\times H(\mathbf{Q})\backslash H(\mathbf{A})} F_\xi(v_{c_1,-c_2})(\iota(h)) F_\omega(h_1) F_\eta(h_2)\, \mathrm{d}h,$$

where $v_{c_1,-c_2}$ is the weight $(c_1,-c_2)$ standard basis vector of τ_2. Note that the integrand is invariant under K_∞. (It is also invariant under a finite-index subgroup of $\mathbf{A}^\times$; in practice we will be interested in the case when ξ, ω, and η have product 1, so we may instead take the integral over $\mathbf{A}^\times H(\mathbf{Q})\backslash H(\mathbf{A})$.)

3.5. **General $t \geqslant 0$.** In general, let us write $c_1' = c_1 + 2t = c_2 - (k_1 - k_2 + 2)$, so that (c_1', c_2) satisfies the assumptions of the previous section. Then the *Maass–Shimura derivative* δ^t sends the holomorphic form F_ω to a non-holomorphic automorphic form $F_\omega^{(t)}$ of K_∞-type $-c_1'$ (i.e. a nearly-holomorphic form of weight c_1' in the sense of Shimura). So we may form the more general integral

$$\langle \xi, \delta^t(\omega) \boxtimes \eta \rangle = \frac{1}{(2\pi i)^2} \int_{\mathbf{R}^\times H(\mathbf{Q})\backslash H(\mathbf{A})} F_\xi(v_{c_1',-c_2})(\iota(h)) \delta^t F_\omega(h_1) F_\eta(h_2)\,\mathrm{d}h,$$

which *a priori* defines a pairing between the groups in (3) after extension to $\mathbf{C}$; our goal is to show that it respects the rational structures.

As in many previous works (e.g. [Urb14, DR14]), we can interpret $F_\omega^{(t)}$ as a section of a larger vector bundle $\widetilde{\mathcal{V}}_{(-c1')} \supseteq \mathcal{V}_{(-c_1')}$, corresponding to a reducible representation of B_{GL_2}. Arguing as in [LPSZ19], we can find a sheaf $\widetilde{\mathcal{V}}^G_{\kappa_2} \twoheadrightarrow \mathcal{V}^G_{\kappa_2}$ (defined as a subquotient of the Hodge filtration on the sheaf attached to the algebraic representation of weight ν), and a pairing $\iota^*\left(\widetilde{\mathcal{V}}^G_{\kappa_2}\right) \times \widetilde{\mathcal{V}}^H_{(-c1',c_2-2)} \to \Omega^1_{X_H}$, which is compatible with the obvious pairing $\iota^*\left(\mathcal{V}^G_{\kappa_2}\right) \times \mathcal{V}^H_{(-c1',c_2-2)} \to \Omega^1_{X_H}$. Moreover, the map on $H^1(X_{G,\mathbf{Q}},-)$ induced by the quotient map $\widetilde{\mathcal{V}}^G_{\kappa_2} \twoheadrightarrow \mathcal{V}^G_{\kappa_2}$ is an isomorphism on the π_{f}-eigenspace. So we can interpret $\langle \xi, \delta^t(\omega) \boxtimes \eta \rangle$ as a cup-product in the cohomology of these larger sheaves; in particular, it respects the E-structures on the cohomology groups, where E is the rationality field of π_{f}.

Remark 3.2. If (c_1, c_2) lies in region (D), but not in the subregion (D^-), then the above construction does not work, because the K^H_∞-type $(c_1', -c_2)$ no longer appears in τ_2. It seems possible that this can be "repaired" by applying differential operators to F_ξ, rather than to the two GL_2 factors; we hope to investigate this further in a future work. ◇

4. Rationality results for Gan–Gross–Prasad periods

Let π be as in the previous section; and let σ_1, σ_2 be cuspidal automorphic representations of GL_2, generated by holomorphic cuspidal modular forms of weights c_1 and c_2 respectively, such that (c_1, c_2) lies in region (D^-) of our diagram. We suppose that ω and η are in the σ_1, resp. σ_2, isotypic part of the cohomology groups.

Theorem 4.1. *If all three classes ξ, ω, η are defined over some number field E, then the period $\langle \xi, \delta^t(\omega) \boxtimes \eta \rangle$ is in E.*

We briefly recall the relation between this period and central L-values. If π and $\sigma_1 \otimes \sigma_2$ have trivial central characters (and thus factor through SO_5 and SO_4 respectively), and the local root numbers $\varepsilon_v(\pi_v \times \sigma_{1,v} \times \sigma_{2,v})$ are $+1$ for all finite places, then the Ichino–Ikeda conjecture [II10] predicts a formula for the absolute value of the global period. This has the form

$$\frac{|\langle \xi, \delta^t(\omega) \boxtimes \eta \rangle|^2}{\|F_\xi\|^2 \cdot \|F_{\delta^t(\omega)\boxtimes\eta}\|^2} = (\star) \cdot \frac{L(\pi \times \sigma_1 \times \sigma_2, \tfrac{1}{2})}{L(\mathrm{ad}\,\pi, 1) L(\mathrm{ad}\,\sigma_1, 1) L(\mathrm{ad}\,\sigma_2, 1)} \prod_v c_v,$$

where $(*)$ is an explicit factor, and c_v are local matrix coefficients (which are nonzero, and equal to 1 for all but finitely many places). So if the Ichino–Ikeda conjecture holds, then Theorem 4.1 determines $L(\pi \times \sigma_1 \times \sigma_2, \frac{1}{2})$ up to an algebraic factor (although making this explicit would involve computing the local matrix coefficient c_v, which is a nontrivial task, particularly for $v = \infty$).

5. Novodvorsky's zeta integral

We now return to the case considered in the introduction, so $\pi \times \sigma$ is an automorphic representation of $\mathrm{GSp}_4 \times \mathrm{GL}_2$ and the weights (k_1, k_2, ℓ) satisfy the inequalities (D) of Table 1. We consider Novodvorsky's zeta integral

$$Z(F_0, \Phi_1, F_2; s) = \int_{Z_H(\mathbf{A}) H(\mathbf{Q})\backslash H(\mathbf{A})} F(h) E^{\Phi_1}(h_1; \chi, s) F'(h_2) \mathrm{d}h,$$

where F and F' are automorphic forms in π and σ respectively, and $E^{\Phi_1}(h_1;\chi,s)$ is an Eisenstein series, depending on a choice of Schwartz function $\Phi_1 \in \mathcal{S}(\mathbf{A}^2)$.

5.1. **Expression via coherent cohomology.** We first show how this zeta integral can be interpreted as a coherent cup product of the type considered in the previous sections, but with ω an Eisenstein, rather than cuspidal, form. We take $F_0 = F_\xi$ and $F_1 = F_\eta$, for coherent cohomology classes ξ, η contributing to $H^1(X_G)$ and $H^1(X_{\mathrm{GL}_2})$, as before (so we are taking $c_2 = \ell$). We suppose $\ell \leqslant k_1$ (so we are in case (D^-)).

As explained in [LPSZ19], for suitable choices of parameters $E^{\Phi_1}(h_1;\chi,s)$ is a nearly-holomorphic Eisenstein series.

We take $\Phi_{1,\infty}(x,y) = 2^{1-c_1'}(x+iy)^{c_1'}\exp(-\pi(x^2+y^2))$, where $c_1' = \ell - (k_1 - k_2 + 2) \geqslant 1$; and let Φ be any Schwartz function of the form $\Phi_{1,\mathrm{f}} \times \Phi_{1,\infty}$, for this particular $\Phi_{1,\infty}$ and any Schwartz function on $\mathbf{A}_{\mathrm{f}}^2$. The condition for s to be a critical value is precisely that $s = \frac{c_1'}{2} \bmod \mathbf{Z}$ and $|2s-1| \leqslant c_1' - 1$; and for s satisfying this, $E^{\Phi_1}(-;\chi,s)$ is nearly-holomorphic (of weight c_1'. Moreover, if we let $c_1 = 1 + |2s-1|$ and $t = \frac{c_1'-c_1}{2} \in \mathbf{Z}_{\geqslant 0}$, then there is a holomorphic Eisenstein series of weight c_1 whose image under δ^t is $E^{\Phi_1}(-;\chi,s)$. (For $\Phi_{1,\mathrm{f}}$ of a certain specific form this is proved in [LLZ14, Corollary 5.2.1]; the general proof is no different.)

Remark 5.1. One can check that $\Phi_{1,\mathrm{f}}$ takes values in our fixed number field E, then $E^{\Phi_1}(-;\chi_1,s)$ is defined over that number field (as a coherent cohomology class). ◇

Corollary 5.2. *If Φ_{f} is E-valued, and ξ, η are defined over E as coherent cohomology classes, then $Z(F_0,\Phi_1,F_2;s)$ lies in $(2\pi i)^2 E$.*

5.2. **Eulerian factorisation.** As explained in [LPSZ19], if π and σ are generic, then we can write this as an integral in terms of the Whittaker functions W_0 and W_2 associated to F_0 and F_2. If the data W_0, Φ_1, W_2 are factorisable as products of local data, then the global integral has a corresponding factorisation as $\prod_v Z_v(W_{0,v},\Phi_{1,v},F_{2,v};s)$, where

$$Z_v(W_{0,v},\Phi_{1,v},F_{2,v};s) = \int_{(Z_H N_H\backslash H)(\mathbf{Q}_v)} W_{0,v}(h) f^{\Phi_{1,v}}(h_1;\chi_v,s) W_{2,v}(h_2)\,\mathrm{d}h.$$

For all but finitely many places, the local integral Z_v is equal to the L-factor $L(\pi_v \times \sigma_v, s)$, so we can write

$$Z(F_0,\Phi_1,F_2;s) = L(\pi\times\sigma,s)\cdot Z_\infty(W_{0,\infty},\Phi_{1,\infty},W_{2,\infty};s)\cdot \prod_{\ell\in S}\frac{Z_v(W_{0,\ell},\Phi_{1,\ell},W_{2,\ell};s)}{L(\pi_\ell\times\sigma_\ell,s)},$$

where S is a finite set of (finite) primes.

For $v = \ell$ a finite prime in S, the ratio $\dfrac{Z_\ell(W_{0,\ell},\Phi_{1,\ell},W_{2,\ell};s)}{L(\pi_\ell\times\sigma_\ell,s)}$ is a rational function in ℓ^{-s} and ℓ^s. The local assumption on $\pi\times\sigma$ in Theorem 1.3 implies that this function is actually a polynomial, and the ideal generated by these polynomials (as $W_{0,\ell},\Phi_{1,\ell},W_{2,\ell}$ vary) is the unit ideal. One can check that if $s = \frac{w}{2} \bmod \mathbf{Z}$, and the data $(W_{0,\ell},\Phi_{1,\ell},W_{2,\ell})$ are defined over E, then the zeta integral is itself E-valued.

If we choose test data of this form at the local places, and equal to the standard Moriyama test data at ∞, then we can conclude that

$$Z(F_0,\Phi_1,F_2;s) \in E^\times \cdot Z_\infty \cdot L(\pi\times\sigma,s)$$

where Z_∞ is the local zeta integral for Moriyama's standard test data at ∞; but we also have

$$Z(F_0,\Phi_1,F_2;s) \in E^\times \cdot (2\pi i)^2 \langle h,h\rangle \Omega_\pi^{(1)}.$$

Thus, in order to prove Theorem 1.3, it remains to compute the Archimedean local integral Z_∞.

6. Archimedean zeta integrals

6.1. **Zeta-integral generalities.** For Π a smooth representation of $\mathrm{GSp}_4(F)$, where F is a local field (archimedean or not), we have the two-parameter $\mathrm{GSp}_4 \times \mathrm{GL}_2$ zeta-integral

$$Z(W,\Phi_1,\Phi_2;\chi_1,\chi_2,s_1,s_2).$$

If we write this in terms of Bessel models, it is

$$\int_{DN_H\backslash H} B_W(h; s_1 - s_2 + \tfrac{1}{2}) f^{\Phi_1}(h_1; s_1, \chi_1) f^{\Phi_2}(h_2; s_2, \chi_2) \mathrm{d}h. \tag{4}$$

We have the Iwasawa decomposition $H = B_H K_H$, where K_H is the maximal compact. If the data are chosen so that the integrand is K-invariant, then (using the fact that the f^Φ's live in principal-series representations, so have a known transformation property under B_H) we obtain

$$Z(W, \Phi_1, \Phi_2, s_1, s_2) = f^{\Phi_1}(1; \chi_1, s_1) f^{\Phi_2}(1; \chi_2, s_2) \int_{F^\times} B_W(\begin{pmatrix} t & & & \\ & t & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; s_1 - s_2 + \tfrac{1}{2}) |t|^{(s_1+s_2-2)} \,\mathrm{d}^\times t.$$

If $F = \mathbf{R}$, then we have an explicit formula for $\int_{F^\times} B_W(\dots)(\dots)$ due to Moriyama, which we recall below.

We want to use this to study $\mathrm{GSp}_4 \times \mathrm{GL}_2$ zeta-integrals. For $c \in \mathbf{Z}_{\geqslant 1}$ there is a (limit-of-)discrete-series representation D_c^+ of $\mathrm{SL}_2(\mathbf{R})$, corresponding to weight c holomorphic modular forms, whose lowest K-type is $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{ic\theta}$. The Whittaker function of the lowest K-type vector is given along the torus by

$$W^{(k)}(\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)) = t^{c/2} e^{-2\pi t}$$

(up to an arbitrary scalar, but if we want our Whittaker functions to match up with the conventional notion of q-expansions then this is clearly the good normalisation).

We can embed D_c^+ into a principal-series representation $I(|\cdot|^{s-1/2}, |\cdot|^{1/2-s}\chi^{-1})$, taking $\chi = (\mathrm{sign})^c$ and $s = \frac{c}{2}$. Then the principal series is reducible, with $D_c^+ \oplus D_c^-$ as a subrepresentation and a finite-dimensional representation as quotient.

If we consider the function $\Phi^{(c)}(x, y) = 2^{1-c}(x + iy)^c e^{-\pi(x^2+y^2)}$, then $f^{\Phi^{(c)}}(-; \mathrm{sign}^c, s)$ has the same K-type, and it lives in $I(|\cdot|^{s-1/2}, |\cdot|^{1/2-s} \mathrm{sign}^c)$. Specializing at $s = \frac{c}{2}$, $f^{\Phi^{(c)}}(-; \mathrm{sgn}^c, \frac{c}{2})$ must land in the D_c^+ subrepresentation, since its K-type does not appear in any of the other factors. Hence, its image under the Whittaker transform, $W^{\Phi^{(c)}}(-, \mathrm{sgn}^c, c/2)$, must be a scalar multiple of the above Whittaker function.

If we evaluate the Whittaker function $W^{\Phi^{(c)}}(-, \mathrm{sgn}^c, c/2)$ at the identity, we are led to a rather nasty definite integral, which eventually turns out to be $e^{-2\pi}$. This is the value at 1 of the normalised Whittaker function above, so our normalisations are compatible (the archimedean analogue of the compatibility noted in [LPSZ19, p4097]). That is, if we substitute $\Phi_2 = \Phi^{(c_2)}$, $s_2 = \frac{c_2}{2}$, and $\chi_2 = \mathrm{sgn}^{c_2}$ in Moriyama's formulae, and let $s = s_1$, we obtain a formula for the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ zeta integral $Z(W, \Phi_1, W^{(c_2)}; s)$.

6.2. Choosing the parameters.

Let (r_1, r_2) be integers with $r_1 \geqslant r_2 \geqslant -1$. This determines an L-packet of representations of $\mathrm{GSp}_4(\mathbf{R})$, which are discrete-series if $r_2 \geqslant 0$ and limit-of-discrete-series if $r_2 = -1$, as usual.

Notation. *In the notations of Moriyama's paper* [Mor04]*, let* $(\lambda_1, \lambda_2) = (r_1 + 3, -1 - r_2)$*; and set* $d = \lambda_1 - \lambda_2 = r_1 + r_2 + 4$.

Then we have the inequalities $1 - \lambda_1 \leqslant \lambda_2 \leqslant 0$ that Moriyama requires (and in fact strict inequality holds). Attached to (λ_1, λ_2), Moriyama defines a pair of discrete / limit-of-discrete series $\mathrm{Sp}_4(\mathbf{R})$-representations $D_{(\lambda_1,\lambda_2)}$ and $D_{(-\lambda_2,-\lambda_1)}$, with $D_{(\lambda_1,\lambda_2)}$ contributing to coherent H^1, and $D_{(-\lambda_2,-\lambda_1)}$ to coherent H^2. Note that the Whittaker functions of $D_{(-\lambda_2,-\lambda_1)}$ are supported on $\mathrm{GSp}_4^+(\mathbf{R})$, while the Whittaker functions of the dual representation are supported on the non-identity component.

We let Π_∞ be the unique representation of $\mathrm{GSp}_4(\mathbf{R})$ whose restriction to $\mathrm{Sp}_4(\mathbf{R})$ is $D_{\lambda_1,\lambda_2} \oplus D_{-\lambda_2,-\lambda_1}$, and whose central character is trivial on $\mathbf{R}_{>0}$.

Remark 6.1. Our parametrisation of the K-types follows [HK92], and unfortunately the conventions of Harris–Kudla and Moriyama are not the same; so in our notations, $D_{(\lambda_1,\lambda_2)}$ has minimal K-type $(-\lambda_2, -\lambda_1)$ and $D_{(-\lambda_2,-\lambda_1)}$ has highest K-type (λ_1, λ_2) (sic!). ◇

6.3. Moriyama's result.

Moriyama defines explicit Whittaker functions W_k, for $0 \leqslant k \leqslant \lambda_1 - \lambda_2$, giving a basis of the minimal K-type of $D_{(-\lambda_2,-\lambda_1)}$. In Proposition 8 of *op.cit.* he states a formula for the integral

$$Z(s, y_1; W) = \int_{y \in \mathbf{R}^\times} \int_{x \in \mathbf{R}} W\left(\begin{pmatrix} uy & & & \\ & y & & \\ & x & 1 & \\ & & & u^{-1} \end{pmatrix}\right) |y|^{s-3/2} \,\mathrm{d}x \,\mathrm{d}^\times y,$$

for $u > 0$ an auxiliary parameter. This involves a Mellin inversion integral, and we get rid of this by taking the forward Mellin integral to get a formula for the Mellin transform of the Bessel function along the torus. (NB: Moriyama uses the other model of GSp_4, with the last two rows & last two columns of the matrix switched.) Unravelling this, we get the following formula: for $0 \leqslant k \leqslant d$, if W_k is the vector of K_H°-type $(-r_1 - 3 + k, r_2 + 1 - k)$, then[2]

$$\int_{\mathbf{R}^\times} B_{W_k}\left(\begin{pmatrix} u & & & \\ & u & & \\ & & 1 & \\ & & & 1 \end{pmatrix}; s_1 - s_2 + \tfrac{1}{2}\right)|u|^{s_1+s_2-2}\,\mathrm{d}u = C \cdot \frac{(-1)^k L(\Pi_\infty, s_1 - s_2 + \frac{1}{2}) L(\Pi_\infty, s_1 + s_2 - \frac{1}{2})}{\pi^{s_1+s_2-\frac{1}{2}}\Gamma(s_1 + \frac{r_1+3-k}{2})\Gamma(s_2 + \frac{-1-r_2+k}{2})},$$

where C is some constant (depending on (r_1, r_2) but not on any of the other data). This is [LPSZ19, Theorem 8.21].

6.4. **Region (F) case revisited.** For region (F), we apply this to compute $Z(W_k, \Phi^{(c_1)}, W^{(c_2)}; s)$ for integers (c_1, c_2) with $c_i \geqslant 1$ and $c_1 + c_2 = r_1 - r_2 + 2$. We take $k = r_1 + 3 - c_1 = r_2 + 1 - c_2$. Then the K-type of W_k is $(-c_1, -c_2)$, meaning it can pair nontrivially with a pair of holomorphic forms of weights c_1 and c_2.

Then we get

$$C \cdot \frac{(-1)^k L(\Pi_\infty, s_1 - s_2 + \frac{1}{2}) L(\Pi_\infty, s_1 + s_2 - \frac{1}{2})}{\pi^{s_1+s_2-\frac{1}{2}}\Gamma(s_1 + \frac{c_1}{2})\Gamma(s_2 + \frac{c_2}{2})} \cdot f^{\Phi^{(c_1)}}(1, s_1) f^{\Phi^{(c_2)}}(1, s_2)\Bigg|_{(s_1,s_2)=(s,\frac{c_2}{2})}.$$

An explicit computation gives

$$f^{\Phi^{(c)}}(1, s) = 2^{1-c} i^c \pi^{-(s+c/2)}\Gamma(s + \tfrac{c}{2}),$$

so up to factors which don't depend on the c_i and hence can be absorbed into C.

Moreover, for region (F) we have

$$L(\Pi_\infty, s_1 - s_2 + \tfrac{1}{2}) L(\Pi_\infty, s_1 + s_2 - \tfrac{1}{2}) = L(\Pi_\infty \times \Sigma_\infty, s).$$

So we get $(\mathrm{const}) \cdot (-1)^{c_2} \cdot L(\Pi_\infty \times \Sigma_\infty, s)$, which is (by definition of critical values) non-zero in the critical range.

6.5. **Region (D).** Now we are going to take c_1, c_2 with $c_1 \geqslant 1$ and $c_2 - c_1 = r_1 - r_2 + 2$; and we choose

$$k = r_1 + 3 + c_1 = c_2 + r_2 + 1.$$

The constraint $k \leqslant d = r_1 + r_2 + 4$ corresponds to $c_2 \leqslant r_1 + 3$, which is the inequality required for the "bottom half" of region (D). Our test data will be

$$Z(W_k, \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) \Phi^{(c_1)}, W^{(c_2)}),$$

and since acting by $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)$ does not change the values of f^Φ along the torus, we can write this as

$$C \cdot \frac{(-1)^k L(\Pi_\infty, s_1 - s_2 + \frac{1}{2}) L(\Pi_\infty, s_1 + s_2 - \frac{1}{2})}{\pi^{s_1+s_2-\frac{1}{2}}\Gamma(s_1 - \frac{c_1}{2})\Gamma(s_2 + \frac{c_2}{2})} \cdot f^{\Phi^{(c_1)}}(1, s_1) f^{\Phi^{(c_2)}}(1, s_2)\Bigg|_{(s_1,s_2)=(s,\frac{c_2}{2})}.$$

Note the change from $s_1 + \frac{c_1}{2}$ to $s_1 - \frac{c_1}{2}$ in the denominator.

On the other hand, in this case the numerator is not $L(\Pi_\infty \times \Sigma_\infty, s)$ any more; one computes that

$$L(\Pi_\infty, s + c_2 - \tfrac{1}{2}) L(\Pi_\infty, s - c_2 + \tfrac{1}{2}) = L(\Pi_\infty \times \Sigma_\infty, s) \cdot (2\pi)^{c_1} \frac{\Gamma(s - \frac{c_1}{2})}{\Gamma(s + \frac{c_1}{2})}.$$

So the zeta-integral computes to

$$(\mathrm{const}) \cdot (-2\pi)^{c_2} \cdot L(\Pi_\infty \times \Sigma_\infty, s),$$

for some constant depending only on (r_1, r_2); and we may choose our normalisation of the Archimedean Whittaker function so that this constant is 1.

[2] Note there is a typo on p. 4108 of [LPSZ19], we erroneously flipped the two components of the K_H°-type. Accordingly, the value of k given there is wrong and should be replaced with $d - k$.

7. Local zeta integrals at p

7.1. **Nonarchimedean L-factors.** We can extend the previous computations to non-archimedean primes. For that purpose, let F be a non-archimedean field, and write $L(\pi \times \sigma, s)$ for the local L-factor associated to $\pi \otimes \sigma$ via Shahidi's method, as in [GT05, §4]. We recall the following result from [LPSZ19, Thm. 8.9].

Proposition 7.1. *The vector space of functions on $\mathbf{C}$ spanned by the $Z(W, \Phi_1, \Phi_2; \chi_1, \chi_2, s_1, s_2)$, as the data (W, Φ_1, Φ_2) vary, is a fractional ideal of $\mathbf{C}[q^{\pm s}]$ containing the constant functions. If at least one of π and σ is unramified, this fractional ideal is generated by the L-factor $L(\pi \otimes \sigma, s)$. Further, there exists a canonical normalization for which the zeta integral is precisely $L(\pi \otimes \sigma, s)$.*

The same considerations presented in [LPSZ19, §8.3.1] regarding the rationality of the Whittaker models also apply.

7.2. **A special H-orbit on FL_G.** From now on, let p be a fixed prime. In this section, we discuss how to compute the local integrals at the prime p for our choice of local conditions, and how this recovers the expected Euler factor.

Lemma 7.2. *Let $\tau = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \in M_{\mathrm{Si}}$, and let $\hat{\tau} = \tau w_1$. Then $H\hat{\tau}P$ is open in G, and $H \cap \hat{\tau}P\hat{\tau}^{-1}$ is a copy of GL_2, embedded in H via*

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right),$$

and in G via

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right) \mapsto \begin{pmatrix} a & b & & b \\ c & d & c & \\ & & a & -b \\ & & -c & d \end{pmatrix}.$$

Proof. Elementary computation. □

Lemma 7.3. *Suppose $\bar{n} \in \overline{N}(\mathbf{Z}_p)$ is congruent to 1 modulo p^k, for $k \geqslant 1$. Then we may write $\hat{\tau}\bar{n} = h\hat{\tau}p$ for some $h \in H(\mathbf{Z}_p)$ and $p \in P(\mathbf{Z}_p)$, with both h and p congruent to 1 modulo p^k.*

Proof. This follows from the matrix identity

$$h(x, y, z)\hat{\tau} \begin{pmatrix} 1 & & & \\ & 1 & & \\ x & y & 1 & \\ z & x & & 1 \end{pmatrix} = \hat{\tau}p(x, y, z),$$

where

$$h(x, y, z) = \left(\begin{pmatrix} (x+1) - \frac{yz}{x+1} & \frac{y}{x+1} \\ -z & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & x+1 \end{pmatrix} \right), \qquad p(x, y, z) = \begin{pmatrix} x+1 & y & 0 & y/(x+1) \\ 0 & x+1 & 0 & 0 \\ & & 1 & -y/(x+1) \\ & & & 1 \end{pmatrix}.$$ □

7.3. **Siegel Jacquet modules.** Now let π be an irreducible smooth generic representation of $\mathrm{GSp}_4(\mathbf{Q}_p)$. We suppose π is is the normalised induction of a representation $\rho \times \lambda$ of $M(\mathbf{Q}_p)$. Note that we have

$$L(\pi, s) = L(\lambda, s)L(\rho \otimes \lambda, s)L(\omega_\rho \lambda, s),$$

and the modulus character δ_P is $(A, c) \mapsto |\det(A)/c|^3$.

Lemma 7.4. *The normalised Jacquet module with respect to $\overline{P}$, $J_{\overline{P}}(\pi) = \pi_{\overline{N}} \otimes \delta_P^{1/2}$, contains a unique subrepresentation isomorphic to $\rho \times \lambda$.*

Proof. Bernstein's second adjointness theorem shows that $\mathrm{Hom}_M(\rho \times \lambda, J_{\overline{P}}(\pi)) = \mathrm{Hom}_G(\pi, \pi) \cong \mathbf{C}$. □

Thus the unnormalised Jacquet module $\pi_{\overline{N}} = \pi/\pi(\overline{N})$ contains a canonical M-subrepresentation isomorphic to $(\rho \times \lambda) \otimes \delta_P^{-1/2}$. We write $\pi[\lambda]$ for the preimage in π of this subrepresentation of $\pi_{\overline{N}}$. For any $v \in \pi[\lambda]$, we have $\mathrm{diag}(1, 1, x, x)v = |x|^{3/2}\lambda(x)v \bmod \pi(\overline{N})$.

Note 7.5. If ρ and λ are unramified, then a vector invariant under the depth t Siegel parahoric $K_{G,\mathrm{Si}}(p^t)$ has this property if and only if it lies in the $U_1' = \alpha$ eigenspace (modulo the zero generalised eigenspace), where $\alpha = p^{3/2}\lambda(p)$ and U_1' is the Hecke operator given by the double coset of $\mathrm{diag}(1, 1, p, p)$. ⋄

7.4. **Trilinear forms and Siegel Jacquet modules.** Let π be as above, and let σ_1, σ_2 be irreducible G-representations with $\omega_\pi \omega_{\sigma_1} \omega_{\sigma_2} = 1$. One knows that $\mathrm{Hom}_{H(\mathbf{Q}_p)}(\pi \times \sigma_1 \times \sigma_2, \mathbf{C})$ has dimension $\leqslant 1$. We suppose it is nonzero, and choose a basis vector $\mathfrak{z}$.

Remark 7.6. If one or more of the σ_i is principal-series, we can construct $\mathfrak{z}$ using Novodvorsky's $\mathrm{GSp}_4 \times \mathrm{GL}_2$ zeta integral. ◇

Proposition 7.7. *For $x \in \pi$, $y_i \in \sigma_i$, we consider the sequence of elements $(z_k(x, y_1, y_2))_{k \geqslant 0}$ defined by*

$$z_k(x, y_1, y_2) = \left(\frac{\lambda(p)}{p^{3/2}}\right)^{-k} \mathfrak{z}\left(\hat{\tau} s_k x, y_1, \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) y_2\right), \qquad s_k = \mathrm{diag}(1, 1, p^k, p^k).$$

If $x \in \pi[\lambda]$, then (z_k) is eventually constant, and its limiting value depends only on the image of x in $\pi[\lambda]/\pi_{\overline{N}}$.

Proof. We first show that if $n \in \overline{N}(\mathbf{Q}_p)$, then we have

$$z_k(\overline{n}x, y_1, y_2) = z_k(x, y_1, y_2) \qquad \forall k \gg 0$$

(for any fixed x, y_1, y_2). Since the elements $s_k \overline{n} s_k^{-1}$ approach the identity as $k \to \infty$, for all $k \gg 0$ we can write

$$\hat{\tau} s_k \overline{n} s_k^{-1} = h_k \hat{\tau} p_k, \qquad h_k \in H(\mathbf{Z}_p), p_k \in P(\mathbf{Z}_p),$$

with h_k and p_k tending to 1 as $k \to \infty$. For all sufficiently large k, h_k will fix $y_1 \otimes \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) y_2$, so we have

$$z_k(\overline{n}x, y_1, y_2) = \left(\frac{\lambda(p)}{p^{3/2}}\right)^{-k} \mathfrak{z}\left(\hat{\tau} s_k \gamma_k x, y_1, \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right) y_2\right), \qquad \gamma_k = s_k^{-1} p_k s_k.$$

Since conjugation by s_k acts trivially on $M(\mathbf{Q}_p)$, and shrinks $N(\mathbf{Q}_p)$, the fact that $p_k \to 1$ certainly implies $\gamma_k \to 1$; so $\gamma_k x = x$ for sufficiently large k. This proves the claim.

Since $z_{k+1}(x, y_1, y_2) = \frac{p^{3/2}}{\lambda(p)} z_k(s_1 x, y_1, y_2)$ by definition, and for $x \in \pi[\lambda]$ we have $s_1 x = \frac{\lambda(p)}{p^{3/2}} x \bmod \pi_{\overline{N}}$, it follows that z_k is eventually constant for $x \in \pi[\lambda]$. □

Definition 7.8. *For $\mathfrak{z} \in \mathrm{Hom}(\pi \times \sigma_1 \times \sigma_2, \mathbf{C})$, we write $\partial_{\mathrm{Si}}(\mathfrak{z})$ for the trilinear form on $\pi[\lambda] \times \sigma_1 \times \sigma_2$ mapping (x, y_1, y_2) to $\lim_{k \to \infty} z_k(x, y_1, y_2)$.*

One checks that for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2$, one has

$$\partial_{\mathrm{Si}}(\mathfrak{z})\left(\begin{pmatrix} A & \\ & \det(A) A' \end{pmatrix} x, A y_1, A y_2\right) = \partial_{\mathrm{Si}}(x, y_1, y_2).$$

So we have defined a map

$$\partial_{\mathrm{Si}} : \mathrm{Hom}_H(\pi \times \sigma_1 \times \sigma_2, \mathbf{C}) \to \mathrm{Hom}_{\mathrm{GL}_2}(\sigma_0 \times \sigma_1 \times \sigma_2, \mathbf{C}), \tag{5}$$

where $\sigma_0 = \rho \otimes \lambda$ is the restriction of $(\rho \times \lambda) \otimes \delta_P^{-1/2}$ to GL_2, embedded in M via $A \mapsto (A, \det(A))$. Note that both source and target of this map have dimension $\leqslant 1$.

Proposition 7.9. *Suppose x, y_1, y_2 are invariant under the principal congruence subgroup modulo p^t for some $t \geqslant 1$, and we have $U_1' x = \alpha x$, where $\alpha = p^{3/2} \lambda(p)$. Then $x \in \pi[\lambda]$, and we have $\mathfrak{z}_k(x, y_1, y_2) = \mathfrak{z}_0(x, y_1, y_2)$ for all $k \geqslant 0$.*

Proof. The relation $U_1' x = \alpha x$ translates into $p^3 s_1 x = \alpha x \bmod \pi(\overline{N})$, since U_1' is the composite of s_1 and a sum over p^3 coset representatives all lying in $\overline{N}$. Thus $x \in \pi[\lambda]$.

A similar argument shows that $\alpha^k \mathfrak{z}_0(x, y_1, y_2) = \mathfrak{z}_0((U_1')^k x, y_1, y_2) = p^{3k} \mathfrak{z}_0(s_k x, y_1, y_2)$, so $z_k(x, y_1, y_2) = z_0(x, y_1, y_2)$. □

7.5. **Relating the zeta-integrals.** We now identify π, σ_1, and σ_2 with their Whittaker models. More precisely, as in [LPSZ19] we take Whittaker models for π with respect to some additive character Ψ, and for σ_1 and σ_2 with respect to Ψ^{-1}.

If π, the σ_i, and Ψ are all unramified, then there is a canonical spherical vector in the Whittaker model of each, and a unique H-invariant trilinear form $\mathfrak{z}^{\mathrm{sph}}$ satisfying $\mathfrak{z}^{\mathrm{sph}}(W_0^{\mathrm{sph}}, W_1^{\mathrm{sph}}, W_2^{\mathrm{sph}}) = 1$. Similarly, there is a spherical GL_2-invariant trilinear form $\mathfrak{y}^{\mathrm{sph}}$ on $\sigma_0 \times \sigma_1 \times \sigma_2$. We identify $J_{\overline{P}}(W(\pi))$ with $W(\sigma_0)$ via mapping the normalised U_1'-eigenvector $W_\alpha^{\prime,\mathrm{Si}}$ to the normalised spherical vector.

Proposition 7.10. *In this unramified setting we have*

$$\partial_{\mathrm{Si}}(\mathfrak{z}^{\mathrm{sph}}) = \Delta \cdot \mathfrak{y}^{\mathrm{sph}}, \qquad \Delta := \frac{p^2}{(p^2-1)} \cdot L(\omega_\rho \lambda \times \sigma_1 \times \sigma_2, \tfrac{1}{2})^{-1}.$$

Note that $\omega_\pi = \lambda^2 \omega_\rho$, so

$$L(\omega_\rho \lambda \times \sigma_1 \times \sigma_2, s) = L(\lambda^\vee \times \sigma_1^\vee \times \sigma_2^\vee, \tfrac{1}{2}).$$

Proof. Since $\mathfrak{y}^{\mathrm{sph}}$ is 1 at the spherical data, and the spherical vector of σ_0 is the image of W_α', it suffices to calculate

$$\partial_{\mathrm{Si}}(\mathfrak{z}^{\mathrm{sph}})\left(W_\alpha^{\prime,\mathrm{Si}}, W_1^{\mathrm{sph}}, W_2^{\mathrm{sph}}\right) = \mathfrak{z}^{\mathrm{sph}}\left(\hat{\tau} W_\alpha^{\prime,\mathrm{Si}}, W_1^{\mathrm{sph}}, \left(\begin{smallmatrix} -1 & \\ & 1\end{smallmatrix}\right) W_2^{\mathrm{sph}}\right) = \mathfrak{z}^{\mathrm{sph}}\left(\hat{\tau} W_\alpha^{\prime,\mathrm{Si}}, W_1^{\mathrm{sph}}, W_2^{\mathrm{sph}}\right).$$

Since $\hat{\tau}$ lies in the same $H(\mathbf{Z}_p)$-orbit on G/P_{Si} as the element ηJ appearing in [LZ20a], we can apply the formulae of *op.cit.* for Iwahori-level Shintani functions to obtain the stated result. In the notation of *op.cit.* for the Hecke parameters, we have

$$L(\lambda^\vee \times \sigma_1^\vee \times \sigma_2^\vee, \tfrac{1}{2}) = \left(1 - \frac{p^2}{\alpha \mathfrak{a}_1 \mathfrak{a}_2}\right)\left(1 - \frac{p^2}{\alpha \mathfrak{a}_1 \mathfrak{b}_2}\right)\left(1 - \frac{p^2}{\alpha \mathfrak{b}_1 \mathfrak{a}_2}\right)\left(1 - \frac{p^2}{\alpha \mathfrak{b}_1 \mathfrak{b}_2}\right). \qquad \square$$

7.6. **Expansion along σ_2.** We now perform a second "reduction along a Jacquet module" argument.

Proposition 7.11. *Let $W_0, W_1 \in \mathcal{W}(\sigma_0), \mathcal{W}(\sigma_1)$, and choose $t \geqslant 1$ such that W_0, W_1 are fixed by $\left(\begin{smallmatrix} 1 & \\ p^t\mathbf{Z}_p & 1\end{smallmatrix}\right)$. Then the value*

$$\mathfrak{y}^{\mathrm{sph}}\left(W_0, W_1, W'_{\mathfrak{a}_2}[\ell]\right)$$

is independent of $\ell \geqslant t$. Its limiting value is equal to the value at $s = \frac{1}{2}$ of the function defined for $\Re(s) \gg 0$ by

$$\Delta_s' \cdot \int_{\mathbf{Q}_p^\times} W_0\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right)\right) W_1\left(\left(\begin{smallmatrix} x & 0 \\ 0 & 1\end{smallmatrix}\right)\right) \tau(x)|x|^{s-1}\,\mathrm{d}x,$$

which has analytic continuation as a polynomial in $p^{\pm s}$; here τ is the unramified character sending x to $p^{-1/2}\mathfrak{b}_2$, and

$$\Delta_s' = \frac{p}{(p+1)} L(\sigma_0 \times \sigma_1 \times \tau, s)^{-1} = \frac{p}{(p+1)}\left(1 - \frac{p^{2-s}}{\beta \mathfrak{a}_1 \mathfrak{a}_2}\right)\left(1 - \frac{p^{2-s}}{\beta \mathfrak{b}_1 \mathfrak{a}_2}\right)\left(1 - \frac{p^{2-s}}{\gamma \mathfrak{a}_1 \mathfrak{a}_2}\right)\left(1 - \frac{p^{2-s}}{\gamma \mathfrak{b}_1 \mathfrak{a}_2}\right).$$

Proof. One can write down an explicit formula for $\mathfrak{y}^{\mathrm{sph}}$ as the leading term of the $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin–Selberg zeta-integral (corresponding to deforming $\mathfrak{a}_2$ and $\mathfrak{b}_2$ to $\mathfrak{a}_2|\cdot|^{1/2-s}$ and $\mathfrak{b}_2|\cdot|^{s-1/2}$):

$$\mathfrak{y}^{\mathrm{sph}}(W_0, W_1, W_2) = \lim_{s \to \frac{1}{2}} \frac{\langle f_2(g;s), R(g;s)\rangle}{L(\sigma_0 \times \sigma_1 \times \tau, s)},$$

where f_2 denotes the Siegel section corresponding to W_2, $\langle -, - \rangle$ denotes integration over $B_G\backslash G \cong \mathbf{P}^1$, and

$$R(g;s) := \int_{\mathbf{Q}_p^\times} W_0\left(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right) g\right) W_1\left(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right) g\right) \tau(x)|x|^{s-1}\,\mathrm{d}^\times x.$$

For the particular choice of W_2 above, f_2 takes the value p^ℓ on the preimage of the identity in $\mathbf{P}^1(\mathbf{Z}/p^\ell)$, and its value is 0 elsewhere. Since the measure of this neighbourhood is $\frac{1}{p^{\ell-1}(p+1)}$, the integral over $\mathbf{P}^1$ is simply $\frac{p}{p+1} R(1)$. $\square$

7.7. **Conclusions.** Note that the product of the two L-factors $\Delta \cdot \Delta'_s|_{s=1/2}$ appearing in Proposition 7.10 and Proposition 7.11 is a degree 8 factor of the degree 16 L-factor $L(\pi \times \sigma_1 \times \sigma_2, \frac{1}{2})$, and thus corresponds to an 8-dimensional direct summand of the 16-dimensional Weil–Deligne representation associated to $\pi \times \sigma_1 \times \sigma_2$; and the 8-dimensional subrepresentation which we obtain is precisely the one giving the "Panchishkin subrepresentation" of the Galois representation when the weights lie in region (D). So the L-factor is the one denoted $\mathcal{E}^{(d)}$ in [LZ20b], that here we call $\mathcal{E}^{(D)}$ to be consistent with our prevoius notation.

Proposition 7.12. *Let $W_1^{\mathrm{dep}} \in \mathcal{W}(\sigma_1)$ be the normalised p-depleted vector (so that $W_1(\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)) = \mathrm{ch}_{\mathbf{Z}_p^\times}(x)$). Then for any $\ell \geqslant 2$ we have*

$$\mathfrak{z}^{\mathrm{sph}}\left(\hat{\tau} W'_{\alpha,\mathrm{Si}}[\ell], W_1^{\mathrm{dep}}, W'_{\mathfrak{a}_2}[\ell]\right) = \frac{p^3}{(p+1)^2(p-1)}\mathcal{E}^{(D)}.$$

In an ideal world (i.e. if we had a comprehensive version of "Siegel-parabolic higher Hida theory" available to us), the above formula would presumably be the right one to use for the interpolation property of our p-adic L-functions. However, for technical reasons we are constrained to work at Iwahori level, so we need a variant of this formula.

Proposition 7.13. *For any $\ell \geqslant 2$ we have*

$$\mathfrak{z}^{\mathrm{sph}}\left(w_{01}^{-1}\tau w_2 \cdot W'^{,\mathrm{Iw}}_{\alpha,\beta}[\ell], \begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} W_1^{\mathrm{dep}}, W_{\mathfrak{a}_2}\right) = \left(\tfrac{p^2}{\beta \mathfrak{b}_2}\right)^t \cdot \frac{p^3}{(p+1)^2(p-1)} \cdot \mathcal{E}^{(D)}.$$

Proof. Since $w_2 = w_1 \cdot w_{\mathrm{Si}}$, where w_{Si} is the long Weyl element of M_{Si}, we can calculate this quantity as

$$\mathfrak{z}^{\mathrm{sph}}\left(\hat{\tau} \cdot w_{\mathrm{Si}} W'^{,\mathrm{Iw}}_{\alpha,\beta}[\ell], \begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} W_1^{\mathrm{dep}}, w W_{\mathfrak{a}_2}\right).$$

Everything in sight is invariant under the principal congruence subgroup mod p^ℓ, so we may apply Proposition 7.9 to express this as

$$\Delta \cdot \mathfrak{y}^{\mathrm{sph}}\left(w \cdot W'_{\beta/p}, \begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} W_1^{\mathrm{dep}}, w W_{\mathfrak{a}_2}\right),$$

where both w's denote $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \in \mathrm{GL}_2$, and $W'_{\gamma/p}$ is a U'-eigenvector of eigenvalue β/p in $\mathcal{W}(\sigma_0)$. We have We have $wW_{\mathfrak{a}_2} = \mathfrak{b}_2^{-\ell}\begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} W'_{\mathfrak{a}_2}[\ell]$; and similarly $w \cdot W'_{\beta/p}[\ell] = \left(\frac{p^2}{\beta}\right)^\ell W_{\beta/p}$. So we obtain

$$\Delta \cdot \left(\tfrac{p^2}{\beta \mathfrak{b}_2}\right)^\ell \mathfrak{y}^{\mathrm{sph}}\left(\begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} \cdot W_{\beta/p}, \begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} W_1^{\mathrm{dep}}, \begin{pmatrix} p^\ell & \\ & 1\end{pmatrix} W'_{\mathfrak{a}_2}[\ell]\right).$$

After cancelling the $\begin{pmatrix} p^\ell & \\ & 1\end{pmatrix}$ terms using the GL_2-equivariance, we can now conclude using Proposition 7.11. □

Remark 7.14. If, in place of W_1^{dep}, we use the Iwahori eigenvector $W_{\mathfrak{a}_2}$, then this corresponds to deleting the degree-one factor $\left(1 - \frac{p^2}{\gamma \mathfrak{b}_1 \mathfrak{a}_2}\right)$ from $\mathcal{E}^{(D)}$. This gives exactly the "greatest common divisor" of $\mathcal{E}^{(D)}$ and $\mathcal{E}^{(E)}$. ◇

References

[Art04] J. Arthur, *Automorphic representations of* GSp(4), Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, 2004, pp. 65–81. MR 2058604. ↑ 3

[BH06] S. Böcherer and B. E. Heim, *Critical values of L-functions on* $\mathrm{GSp}_2 \times \mathrm{GL}_2$, Math. Z. **254** (2006), no. 3, 485–503. MR 2244361. ↑ 3, 4

[DR14] H. Darmon and V. Rotger, *Diagonal cycles and Euler systems I: a p-adic Gross–Zagier formula*, Ann. Sci. École Norm. Sup. (4) **47** (2014), no. 4, 779–832. MR 3250064. ↑ 8

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and L-functions (Corvallis, 1977), Proc. Sympos. Pure Math., vol. 33.2, Amer. Math. Soc., Providence, RI, 1979, with an appendix by N. Koblitz and A. Ogus, pp. 313–346. MR 546622. ↑ 3

[Emo19] M. Emory, *On the global Gan–Gross–Prasad conjecture for general spin groups*, preprint, 2019, `arXiv:1901.01746`. ↑ 2

[GT18] T. Gee and O. Taïbi, *Arthur's multiplicity formula for* GSp_4 *and restriction to* Sp_4, preprint, 2018, `arXiv:1807.03988`. ↑ 3

[GT05] A. Genestier and J. Tilouine, *Systèmes de Taylor–Wiles pour* GSp_4, Formes automorphes. II. Le cas du groupe GSp(4), Astérisque, vol. 302, Soc. Math. France, 2005, pp. 177–290. MR 2234862. ↑ 12

[Har04] M. Harris, *Occult period invariants and critical values of the degree four L-function of* GSp(4), Contributions to automorphic forms, geometry, and number theory (in honour of J. Shalika), Johns Hopkins Univ. Press, 2004, pp. 331–354. MR 2058613. ↑ 3

[HK92] M. Harris and S. Kudla, *Arithmetic automorphic forms for the nonholomorphic discrete series of* GSp(2), Duke Math. J. **66** (1992), no. 1, 59–121. MR 1159432. ↑ 7, 10

[II10] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425. MR 2585578. ↑ 2, 8

[Lan19] K.-W. Lan, *Closed immersions of toroidal compactifications of Shimura varieties*, preprint, 2019. ↑ 5

[LLZ14] A. Lei, D. Loeffler, and S. L. Zerbes, *Euler systems for Rankin–Selberg convolutions of modular forms*, Ann. of Math. (2) **180** (2014), no. 2, 653–771. MR 3224721. ↑ 9

[LPSZ19] D. Loeffler, V. Pilloni, C. Skinner, and S. L. Zerbes, *Higher Hida theory and p-adic L-functions for* GSp(4), preprint, 2019, `arXiv:1905.08779`. ↑ 3, 5, 7, 8, 9, 10, 11, 12, 14

[LR23] D. Loeffler and O. Rivero, *On p-adic L-functions for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, preprint, 2023. ↑ 4

[LZ20a] D. Loeffler and S. L. Zerbes, *On some zeta-integrals for unramified representations of* GSp(4), in preparation, 2020. ↑ 14

[LZ20b] ———, *p-adic L-functions and diagonal cycles for* $\mathrm{GSp}(4) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$, in preparation, 2020. ↑ 1, 2, 15

[LZ21] ———, *On the Bloch–Kato conjecture for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, preprint, 2021, `arXiv:2106.14511`. ↑ 5, 6

[Mor04] T. Moriyama, *Entireness of the spinor L-functions for certain generic cusp forms on* GSp(2), Amer. J. Math. **126** (2004), no. 4, 899–920. MR 2075487. ↑ 10

[Nov79] M. E. Novodvorsky, *Automorphic L-functions for the symplectic group* GSp(4), Automorphic forms, representations and L-functions (Corvallis, 1977), Proc. Sympos. Pure Math., vol. 33.2, Amer. Math. Soc., 1979, pp. 87–95. MR 546610. ↑ 2

[Sah10] A. Saha, *Pullbacks of Eisenstein series from* GU(3, 3) *and critical L-values for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, Pacific J. Math. **246** (2010), no. 2, 435–486. MR 2652263. ↑ 3, 4

[Su19] J. Su, *Coherent cohomology of Shimura varieties and automorphic forms*, preprint, 2019. ↑ 7

[Urb14] E. Urban, *Nearly overconvergent modular forms*, Iwasawa Theory 2012: State of the Art and Recent Advances (Berlin), Contrib. Math. Comput. Sci., vol. 7, Springer, 2014, pp. 401–441. ↑ 8

[Yos01] H. Yoshida, *Motives and Siegel modular forms*, Amer. J. Math. **123** (2001), no. 6, 1171–1197. MR 1867315. ↑ 3

D.L.: Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom
Email address: `D.A.Loeffler@warwick.ac.uk`

O.R.: Mathematical Sciences Research Institute / Simons Laufer Mathematical Sciences Institute, Berkeley, CA 94720, USA
Email address: `Oscar.Rivero-Salgado@warwick.ac.uk`