DERIVATIVES OF BEILINSON–FLACH CLASSES, GROSS–STARK FORMULAS AND A p-ADIC HARRIS–VENKATESH CONJECTURE

ÓSCAR RIVERO

Abstract. We propose an alternative approach to the study of exceptional zeros from the point of view of Euler systems. As a first application, we give a new proof of a conjecture of Darmon, Lauder and Rotger regarding the computation of the $\mathcal{L}$-invariant of the adjoint of a weight one modular form in terms of units and p-units. While in our previous work with Rotger the essential ingredient was the use of Galois deformations techniques, we discuss a new method exclusively using the properties of Beilinson–Flach classes. One of the key ingredients is the computation of a cyclotomic derivative of a cohomology class in the framework of Perrin-Riou theory, which can be seen as a counterpart to the earlier work of Loeffler, Venjakob and Zerbes. In our second application, we illustrate how these techniques could lead to a better understanding of this setting by introducing a new motivic p-adic L-function whose special values encode information just about the unit of the adjoint (and not also the p-unit), in the spirit of the conjectures of Harris and Venkatesh. We further discuss conjectural connections with the arithmetic of triple products of Coleman families.

Contents

1. Introduction

In our series of works with Rotger [RiRo1], [RiRo2], we proposed a systematic study of the conjecture of Darmon, Lauder and Rotger [DLR] on p-adic iterated integrals in terms of certain cohomology classes constructed from the p-adic interpolation of Beilinson–Flach

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elements. This conjecture may be subsumed in a broader programme comprising both the Gross–Stark conjectures and also the celebrated *Elliptic Stark Conjectures*, which shed some light on the arithmetic of elliptic curves of rank 2. We devote part of this introduction to recall these conjectures in order to put our results in this broad scenario, where the phenomenon of exceptional zeros plays a prominent role.

1.1. **General set up and Beilinson–Flach classes.**

Let χ be a Dirichlet character of level $N \geq 1$, and let $S_1(N,\chi)$ stand for the space of cuspidal modular forms of weight 1, level N and nebentypus χ. Let $g = \sum_{n\geq 1} a_n q^n \in S_1(N,\chi)$ be a normalized newform and let $g^* = g \otimes \bar{\chi}$ denote its twist by the inverse of its nebentypus. Let

$$\varrho_g : \mathrm{Gal}\,(H_g/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V_g) \simeq \mathrm{GL}_2(L), \quad \varrho_{\mathrm{ad}^0(g)} : \mathrm{Gal}\,(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathrm{ad}^0(g)) \simeq \mathrm{GL}_3(L)$$

denote the Artin representations associated with g and its adjoint, respectively. Here $H_g \supseteq H$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by these representations, and L is a finite extension of $\mathbb{Q}$ containing their traces and the roots of the p-th Hecke polynomial of g.

Let $p \nmid N$ be a prime number. Fix once and for all an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, singling out a place of H (resp. L) above p, and a decomposition group $G_{\mathbb{Q}_p}$.

Label and order the roots of the p-th Hecke polynomial of g as $X^2 - a_p(g)X + \chi(p) = (X-\alpha)(X-\beta)$. We assume throughout that

(H1) The reduction of ϱ_g mod p is irreducible;
(H2) g is p-distinguished, i.e. $\alpha \neq \beta \,(\mathrm{mod}\, p)$, and
(H3) ϱ_g is not induced from a character of a real quadratic field in which p splits.

The purpose of (H1) and (H2) is to avoid technical complications regarding the use of Hida families (see e.g. Wiles' result in the form of [KLZ1, Theorem 7.2.8]), although the weaker condition $\alpha \neq \beta$ is crucially used at several points and its failure is indeed quite problematic (as discussed in [DLR2, Part B]). The role of (H3) is of a different nature, and it excludes a qualitatively different situation where the eigencure is not étale, as it was established by Bellaïche and Dimitrov [BeDi].

In [RiRo2], the authors used the theory of Beilinson–Flach elements developed by Kings, Lei, Loeffler and Zerbes to construct *four* (a priori distinct) cohomology classes

$$\kappa(g_\alpha, g^*_{1/\alpha}), \quad \kappa(g_\beta, g^*_{1/\beta}), \quad \kappa(g_\alpha, g^*_{1/\beta}), \quad \kappa(g_\beta, g^*_{1/\alpha}) \in H^1(\mathbb{Q}, \mathrm{ad}^0(g)(1)), \tag{1}$$

arising by considering the different Hida families passing through the pair (g, g^*). This allowed us to reformulate the Gross–Stark conjecture of [DLR], expressing the previous classes in terms of canonical units and p-units in $(\mathcal{O}_H^\times[1/p] \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ and supplying an alternative framework for understanding the results of [DLR], [DR2] and [KLZ1].

As we recall in Proposition 3.1, it happens that

$$\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0 \tag{2}$$

due to an exceptional zero phenomenon which can be explained by the vanishing of an Euler factor.

This is quite unfortunate for our purposes of understanding the Gross–Stark conjecture of [DLR] by resorting to the theory of Euler systems, since there is not a *canonical* cohomology class to be used for such a purpose. To overcome this situation, in our previous works we had constructed certain derivatives of those classes, but it turns out that the definition we had used was not useful to prove the Gross–Stark conjecture or to obtain new results in that direction. Roughly speaking, we had taken the derivative along one of the weight directions associated with the Hida family interpolating one of the modular forms, while towards obtaining a more flexible and arithmetically interesting setting we need to consider also the *cyclotomic* derivative.

This is an analogous situation to the scenario of [Buy2] and [Ven], where the computation of the derivative of the Mazur–Kitagawa p-adic L-function along a certain direction of the weight space was relatively easy using the classical theory of Heegner points (and had already been carried out by Bertolini and Darmon [BD]), but the computation of the *cyclotomic* derivative required new ideas. Hence, this work may be thought as a counterpart to the approach of Büyükboduk and Venerucci to the exceptional zero phenomenon, but in the easier case where elliptic curves are replaced by unit groups (and hence one can circumvent the technical complications introduced by the use of Nekovar's height theory). Similar results had been obtained by Loeffler, Venjakob and Zerbes [LVZ], and one can see our computations as the *dual* of Proposition 2.5.5 and Theorem 3.1.2 of *loc. cit.* We refer also to the seminal works of Benois [Ben1], [Ben2] where similar questions are addressed.

1.2. A new proof of a Gross–Stark formula.

Our main result in [RiRo1] was the computation of a special value formula for the Hida–Rankin p-adic L-function at weight one (alternatively, the derivative of the adjoint of the modular form). This is specially intriguing since that function, that we denote as $L_p(g,g^*,s)$, cannot be directly defined in terms of an interpolation property, and requires to consider the p-adic variation of the modular forms (g,g^*) along a Hida family. Indeed, it depends on the choice of a p-stabilization for g. We sometimes write $L_p^{g_\alpha}(g,g^*,s)$ to emphasize this dependence. The computation of $L_p(g,g^*,1)$ may be better understood in the general framework of the p-adic Gross–Stark conjectures initiated by Gross [Gross], and which predicts that the special values of the p-adic L-function of an Artin representation must encode information about the arithmetic of the number field cut out by it. For our further use along the article, recall also that the functional equation presented in [Das, Section 9.2] implies that $L_p(g,g^*,0) = L_p(g,g^*,1)$ modulo $L^\times$. As the numerical computations of [DLR] suggest, this value is expected to be generically non-zero, and we make the following assumption throughout the article.

(H4) $L_p^{g_\alpha}(g,g^*,1) \neq 0$.

Further, and as a rather straightforward application of Dirichlet's unit theorem combined with Frobenius reciprocity, in [DLR, Section 1] it is shown that

$$\dim_L(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2.$$

Fix a generator u of $(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ and also an element v of $(\mathcal{O}_H^\times[1/p]^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ in such a way that $\{u,v\}$ is a basis of $(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$. The element v may be chosen to have p-adic valuation $\mathrm{ord}_p(v)=1$, and we do so.

Viewed as a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g) = L \oplus L^{\alpha\otimes\bar\beta} \oplus L^{\beta\otimes\bar\alpha}$, where each line is characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β and β/α, respectively. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$, $\hat H_p^\times$ the completion of $H_p^\times$, and let

$$u_1,\ u_{\alpha\otimes\bar\beta},\ u_{\beta\otimes\bar\alpha},\ v_1,\ v_{\alpha\otimes\bar\beta},\ v_{\beta\otimes\bar\alpha} \in \hat H_p^\times \otimes_\mathbb{Q} L \pmod{L^\times}$$

denote the projection of the elements u and v in $(\hat H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines.

Then, we have the following Gross–Stark formula:

Theorem 1.1. *Assume that hypotheses (H1)-(H4) hold. Then, the following equality holds up to multiplication by a scalar in* $L^\times$

$$L_p^{g_\alpha}(g,g^*,1) = \frac{\log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1) - \log_p(v_{\alpha\otimes\bar\beta})\log_p(u_1)}{\log_p(u_{\alpha\otimes\bar\beta})}.$$

The non-vanishing assumption (H4) is somewhat irritating, but it was also present in our previous works when dealing with Beilinson–Flach classes.

The proof we had given in [RiRo1, Section 4] was lengthy and made use of the results of Bellaiche–Dimitrov [BeDi] computing the tangent space of a deformation problem and following the further development of [BDP], together with some elements taken from the earlier work [DLR2]. In a certain way, that proof mimicked the approach of Greenberg–Stevens [GS] to the exceptional zero phenomenon for elliptic curves with split multiplicative reduction. However, the authors had observed a tantalizing connection with the theory of Beilinson–Flach elements, that were affected by a similar exceptional zero phenomenon. This allowed us to interpret *derived* classes of Beilinson–Flach elements in terms of the units $\{u, v\}$, but does not give any new insight into the proof of Theorem 1.1. This work may be seen as a culmination of the purpose that the authors had when they began to write both [RiRo1] and [RiRo2], that was proving the Gross–Stark conjecture of Darmon, Lauder and Rotger using just the properties of Beilinson–Flach elements and the flexibility provided by the notion of *derivatives*. This is just another instance of the power of Euler systems when dealing with arithmetic questions.

We can give, with these ideas at hand, a different proof of the main theorem of [RiRo1]. This can be seen as the counterpart to the approach of Kobayashi [Ko] to the Mazur–Tate–Teitelbaum conjecture in rank 0, since he reproves the result of Greenberg and Stevens using the properties of Kato's cohomology classes.

Our proof is a combination of four main ideas (together with the same starting point coming from Hida's theory of improved p-adic L-functions):

(0) The results of Hida [Hi1], [Hi2], which reduce the conjecture to the computation of the derivative of the Frobenius eigenvalue along the weight direction. This part is common to our earlier work [RiRo1].

(1) The local properties at p of Beilinson–Flach elements, which give an expression, *up to multiplication by a p-adic scalar*, for the derived class $\kappa'(g_\alpha, g_{1/\beta}^*)$ in terms of the units u and v, where here the derivative is taken along any arbitrary direction of the weight space.

(2) A comparison between the different reciprocity laws and the observation that knowing two *weight* derivatives, together with the vanishing of the class $\kappa(\mathbf{g}, \mathbf{g}^*)$ along the line $(\ell, \ell, \ell-1)$, allows us to determine the cyclotomic derivative of the class.

(3) An explicit reciprocity law for the Λ-adic class $\kappa(\mathbf{g}, \mathbf{g}^*)$, obtained when g and g^* vary over Hida families $\mathbf{g}$ and $\mathbf{g}^*$, respectively. This was first proved by Kings–Loeffler–Zerbes [KLZ1]. In our situation, there is an exceptional vanishing, and hence we may consider a *derived* reciprocity law, in the sense of [RiRo1]. This gives an expression for the *weight derivative* of the Beilinson–Flach class in terms of an unknown p-adic period and involving also the $\mathcal{L}$-invariant of the adjoint of g_α.

(4) The results of Büyükboduk [Buy1], [Buy2] and Venerucci [Ven] around Coleman maps, which allow us to relate the *cyclotomic derivative* of the Beilinson–Flach class to the Hida–Rankin p-adic L-function. This part can be also understood, by duality, in terms of the computations developed in [LVZ]. Comparing this result with (3), we get a formula for the $\mathcal{L}$-invariant, and consequently for the special p-adic L-value.

Observe that the study of universal norms has also allowed Roset, Rotger and Vatsal [RRV] to reinterpret the $\mathcal{L}$-invariant of Theorem 1.1 in terms of an algebraic avatar initially defined by Greenberg [Gr].

1.3. **A p-adic version of the Harris–Venkatesh conjecture.** Theorem 1.1 is not completely satisfactory towards the understanding of the arithmetic of the adjoint of a weight one modular form, since it involves both the unit and the p-unit attached to the Galois representation. It is then natural to expect a putative refinement of the previous result in the spirit of the conjectures of Harris–Venkatesh [HV], with a p-adic L-function whose special values encode information just about the unit u. Unless otherwise specified, we keep assumptions (H1)-(H3).

To make the analogy more precise, and following the notations of [DLR], let $E_2 \in M_2(N)$ be the weight two Eisenstein series, and let $F := d^{-1}E_2 = E_0^{[p]}$ be the overconvergent Eisenstein series of weight zero whose Fourier expansion is given by

$$F(q) = \sum_{p\nmid n} \Big(\sum_{d|n} d^{-1} \Big) q^n.$$

Similarly, let

$$\Xi(g_\alpha, g^*) = e_{g_\alpha^*} e_{\mathrm{ord}}(Fg^*),$$

where

$$e_{\mathrm{ord}} \, : \, M_1^{\mathrm{oc}}(N, \bar\chi) \longrightarrow M_1^{\mathrm{oc}}(N, \bar\chi)$$

is Hida's ordinary projection, and

$$e_{g_\alpha^*} \, : \, M_1^{\mathrm{oc}}(N, \bar\chi) \longrightarrow M_1^{\mathrm{oc}}(N, \bar\chi)[[g_\alpha^*]]$$

is the Hecke equivariant projection to the generalised eigenspace attached to g_α^*. Assuming (H3), it consists entirely of classical forms. In particular, as it is discussed in the introduction of [DLR], there is an isomorphism of $\mathbb{C}_p$-vector spaces $M_1(Np, \bar\chi)[g_\alpha^*] \simeq S_1^{\mathrm{oc,ord}}(N, \bar\chi)[[g_\alpha^*]]$.

Let γ be an element in the L-linear dual space $M_1(N, \bar\chi)_L[g_\alpha^*]^\vee$. Then, the value we have computed is an L-multiple of $\gamma(\Xi(g_\alpha, g^*))$, i.e.,

$$\gamma(\Xi(g_\alpha, g^*)) = \frac{\log_p(u_{\alpha\otimes\bar\beta}) \log_p(v_1) - \log_p(v_{\alpha\otimes\bar\beta}) \log_p(u_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times}. \tag{3}$$

Roughly speaking, the quantity $\gamma(\Xi(g_\alpha, g^*))$ may be seen as a pairing in weight one between g and Fg^*. In the study of the arithmetic of triple products, when instead of taking the weight two Eisenstein series E_2 we begin with a cusp form f, there is an alternative pairing in weight two between f and gg^*. This is, up to multiplication by the p-adic adjoint L-function, the value of the triple product p-adic L-function $\mathscr{L}_p{}^f(\mathbf{f}, g, g^*)$, where $\mathbf{f}$ is the unique ordinary Hida family going through f. When f is Eisenstein, however, this value may be recast as the projection of a cuspidal form onto an Eisenstein eigenspace, which is zero. Then, the only natural invariant to consider seemed to be the one described in (3), where this trivial vanishing does not arise.

Harris and Venkatesh, however, proposed a related conjecture modulo p, working in a much broader setting. They construct an element in the dual of $S_2(N, \mathbb{Z}/p\mathbb{Z})$, the *Shimura class*, arising from the étale covering of modular curves $X_1(N) \longrightarrow X_0(N)$. Then, they pair it with a suitable modification of gg^* and conjecture a precise modulo p relation with the logarithm of the unit u. This conjecture has been established under some mild assumption when g is dihedral in a recent work of Darmon, Harris, Rotger and Venkatesh [DHRV].

There seems to be no natural p-adic analogue, although motivated by the recent work of Benois and Büyükboduk [BeBu], we can try to look at the following invariant. Let $\mathbf{f}_{\mathrm{crit}}$ be the Coleman family passing through the critical p-stabilization of the weight two Eisenstein series, that we write E_2^{crit}, and indexed by a weight variable x. Let x_0 be the weight two point such that $\mathbf{f}_{\mathrm{crit},x_0} = E_2^{\mathrm{crit}}$. Then, define the special value

$$\delta_1 = \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, g_{1/\alpha}^*)(x_0).$$

Question. *Triple product L-function with dominant Eisenstein series.*

(a) Does there exist an analytic p-adic L-function $L_p^{\mathrm{Eis}}(g, g^*, s)$ such that

$$L_p^{\mathrm{Eis}}(g, g^*, 0) = \log_p(u_1) \pmod{L^\times}?$$

(b) Can we interpret $L_p^{\mathrm{Eis}}(g, g^*, s)$ in terms of the arithmetic of triple products of the form $(\mathbf{f}_{\mathrm{crit}}, g, g^*)$, where $\mathbf{f}_{\mathrm{crit}}$ is a Coleman family passing through a critical Eisenstein series? Is it true that
$$\delta_1 = \log_p(u_1) \pmod{L^\times}?$$

The last section of this note is devoted to discuss the following conjecture in the framework provided by Perrin-Riou maps, emphasizing the connection with critical Eisenstein series, further explored in forthcoming work with Loeffler. As a last piece of notation, let $\mathcal{L}_{g_\alpha}$ denote the p-adic invariant introduced in [DR2, Section 2]. In *loc. cit.* this invariant is conjectured to be equal to $\log_p(u_{\alpha\otimes\bar\beta})$.

Theorem 1.2. *There exists a motivic p-adic L-function $L_p^{\mathrm{mot}}(g, g^*, s)$, obtained after applying a suitable Perrin-Riou map to the cyclotomic class whose bottom layer is $\kappa(g_\alpha, g^*_{1/\alpha})$, and such that*
$$L_p^{\mathrm{mot}}(g, g^*, 0) = \frac{\mathcal{L}_{g_\alpha}}{\log_p(u_{\alpha\otimes\bar\beta})} \times \log_p(u_1) \pmod{L^\times}.$$

We finish this preliminary discussion by pointing out that the formalism we discuss regarding exceptional zeros and derivatives at the level of Euler systems suggests the possibility of developing an axiomatic treatment mimicking the general conjectures of Greenberg and Benois. More precisely, there are two scenarios where our methods can potentially yield to interesting results.

(I) The work of Betina and Dimitrov [BeDi, Section 4], following ideas around the geometry of the eigencurve already introduced in their joint work with Pozzi [BDP], suggests various formulas for the derivative of the Katz's two variable p-adic L-function along different directions of the weight space. It was already implicit in previous work of Büyukboduk [Buy1] and the author [Ri] that derivatives of the cohomology classes coming from elliptic units, taken along different directions, capture either the p-adic logarithm or the p-adic valuation of the class. The application of this formalism (and its eventual generalizations to CM fields) seems to have interesting applications we hope to explore in the future.

(II) The study of these methods seems specially intriguing in the scenario of "diagonal cycles" or in the novel setting of $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ explored by Loeffler and Zerbes [LZ2], where a cohomology class may simultaneously encode information about the behavior of different, and a priori unrelated, p-adic L-functions. The use of our techniques seem to be able to partially recover Rosso's results [Ros], and to be further extended to other settings.

1.4. **Organization of the note.** The organization of this note is as follows. Section 2 discusses the motivational case of circular units, where these same phenomena arise and that can serve as a motivation for our later work. Section 3 recalls the notations and results of [RiRo1] around Beilinson–Flach elements which are needed in the proof. Section 4 contains the first main result of the article and discuss the new proof of Theorem 1.1 using the notion of *derived* Belinson–Flach elements. Next, Section 5 proposes an alternative interpretation of the previous results in terms of deformation theory, which can help to give the reader a broader picture of this scenario. Finally, Section 6 discusses the p-adic Harris–Venkatesh conjecture we have suggested, introducing the relevant motivic p-adic L-function and analyzing its relationship with the arithmetic of triple products.

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2. Analogy with the case of circular units

The situation we want to deal with has a clear parallelism with the case of circular units, that we now recall. Fix a primitive Dirichlet character χ of conductor N, and write $H = \mathbb{Q}(\zeta_N)$, where ζ_N is a (fixed) primitive N-th root of unity. Let L_χ be its coefficient field. As a piece of notation, $\bar{\chi} = \chi^{-1}$ denotes the character obtained after composing χ with complex conjugation. As in the introduction, let $p \nmid N$ be a prime number. Our choice of an embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ singles out a place of H (resp. L_χ) above p, and a decomposition group $G_{\mathbb{Q}_p}$. We write $L_{\chi,p}$ for the completion of L_χ at the prime p.

The case where χ is odd gives rise to an exceptional vanishing of the Kubota–Leopoldt p-adic L-function $L_p(\chi\omega, s)$ at $s = 0$ when $\chi(p) = 1$, where ω is the Teichmüller character. Under the assumption that χ is odd, $(\mathcal{O}_H^\times \otimes L_\chi)^\chi$ is a zero-dimensional L_χ-vector space, while $(\mathcal{O}_H[1/p]^\times \otimes L_\chi)^\chi$ has dimension 1 if $\chi(p) = 1$. In this case, choosing a non-zero element v_χ of the latter space, we may define an $\mathcal{L}$-invariant

$$\mathcal{L}(\chi) = -\frac{\log_p(v_\chi)}{\mathrm{ord}_p(v_\chi)}, \tag{4}$$

which depends on our choice of p-adic embedding ι_p. Then,

$$L_p'(\chi\omega, s) = \mathcal{L}(\chi) \cdot L(\chi, 0). \tag{5}$$

This formula was firstly established by Gross [Gross], using the Gross–Koblitz formula [GK] and the earlier theorem of Ferrero–Greenberg [FG], which relates the derivative of the Kubota–Leopoldt p-adic L-function to special values of the p-adic Gamma function. This was extended later on in different works by Darmon, Dasgupta, Kakde, Pollack, and Ventullo [DDP], [DKV] to the setting of totally real number fields, using related ideas in the realm of exceptional zeros.

In the case where χ is even, the situation is ostensibly different. Here, $(\mathcal{O}_H^\times \otimes L_\chi)^\chi$ is one-dimensional and we may fix a generator c_χ of it, that we call the *circular* unit associated with χ. We take it, as usual, as a weighted combination of cyclotomic units

$$c_\chi = \prod_{a=1}^{N-1} (1 - \zeta_N^a)^{\chi^{-1}(a)},$$

where the notation $(1 - \zeta_N^a)^{\chi^{-1}(a)}$ means $(1 - \zeta_N^a) \otimes \chi^{-1}(a)$. Moreover, if we further assume that $\chi(p) = 1$, $(\mathcal{O}_H[1/p]^\times \otimes L_\chi)^\chi$ has dimension 2, and we may consider a basis of the form $\{c_\chi, v_\chi\}$, with the convention that $\mathrm{ord}_p(v_\chi) = 1$.

Given any even, non-trivial and primitive Dirichlet character, one always has Leopoldt's formula, which asserts that

$$L_p(\chi, 1) = -\frac{(1 - \chi(p)p^{-1})}{\mathfrak{g}(\bar{\chi})} \cdot \log_p(c_\chi). \tag{6}$$

We may understand the previous result in the more general setting of reciprocity laws. For that purpose, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. A remarkable feature of the unit c_χ is that it can be understood as a *universal norm* over the cyclotomic tower; in particular, fixing a sequence of primitive Np^n-th roots of unity compatible under the p-power map, $(\zeta_N, \zeta_{Np}, \ldots, \zeta_{Np^n}, \ldots)$, we may define

$$c_{\chi,n} = \prod_{a=1}^{N-1} (1 - \zeta_{Np^n}^a)^{\chi^{-1}(a)}.$$

Taking the inverse limits of their images under the Kummer map, one may construct a Λ-adic class

$$\kappa(\chi, s) \in H^1(\mathbb{Q}, L_{\chi,p}(\bar\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}})),$$

where $\varepsilon_{\mathrm{cyc}}$ is the usual cyclotomic character and $\underline{\varepsilon}_{\mathrm{cyc}}$ stands for the Λ-adic cyclotomic character. Given $s \in \mathbb{Z}$, let $\nu_s : \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}) \to \mathbb{Z}_p$ be the ring homomorphism sending the group-like element $a \in \mathbb{Z}_p^\times$ to a^s. This induces a $G_{\mathbb{Q}}$-equivariant specialization map

$$\nu_s : \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}) \to \mathbb{Z}_p(s)$$

and gives rise to a collection of global cohomology classes

$$\kappa(\chi, s) \in H^1(\mathbb{Q}, L_{\chi,p}(\bar\chi)(s)).$$

The Perrin-Riou formalism allows us to understand the Kubota–Leopoldt p-adic L-function $L_p(\chi, s)$ as the image under a Coleman map (also named as Perrin-Riou map, or Perrin-Riou regulator) of the local class $\kappa_p(\chi, s)$

$$\mathcal{L}_\chi : H^1(\mathbb{Q}_p, L_{\chi,p}(\bar\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow I^{-1}\Lambda, \quad \mathcal{L}_\chi(\kappa_p(\chi, s)) = L_p(\chi, s),$$

where I is the ideal of Λ corresponding to the specialization at $s = 1$ (see [KLZ1, Section 8.2] for the precise definition). This map interpolates either the dual exponential map (for $s \leq 0$) or the Bloch–Kato logarithm (for $s \geq 1$).

A standard computation using the explicit definition of the system of units $(c_{\chi,n})$ shows that the bottom layer $\kappa(\chi, 1)$ vanishes when $\chi(p) = 1$. We keep this assumption throughout the rest of this section, pointing out that this kind of exceptional zeros at the level of cohomology classes will be our main object of study along this note. Following the construction of [Buy1, Section 3], there is a derived class $\kappa'(\chi, s)$, defined as the unique class satisfying that

$$\kappa(\chi, s) = \frac{\gamma - 1}{\log_p(\gamma)} \cdot \kappa'(\chi, s), \tag{7}$$

where γ is a fixed topological generator of $\mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. It is also proved in [Buy1] that $\kappa'(\chi, 1)$ belongs to an extended Selmer group, which in this case may be identified with the group of p-units where the Galois group acts via χ (we insist that when χ is even this space is two-dimensional). Hence, the exceptional zero phenomenon does not appear at the level of p-adic L-functions, since at least generically $L_p(\chi, 1) \neq 0$, but at the level of cohomology classes.

Moreover, in the cases where $\chi(p) = 1$ one can also define an *improved* map

$$\widetilde{\mathcal{L}_\chi} = \frac{\gamma - 1}{\frac{1}{p}\log_p(\gamma)} \times \mathcal{L}_\chi : H^1(\mathbb{Q}_p, L_{\chi,p}(\bar\chi)(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow I^{-1}\Lambda.$$

Therefore,

$$\widetilde{\mathcal{L}_\chi}(\kappa_p'(\chi, s)) = p \cdot L_p(\chi, s).$$

The computations done in [Buy1, Section 6.2], in particular Remark 6.5, show that the map $\widetilde{\mathcal{L}_\chi}$, when specialized at $s = 1$, is given by the order map (applied in this case to the derived class). The key point is a computation of the universal norms over the cyclotomic tower, as well as the use of Lemma 6.4 of *loc. cit* (see also [Ven, Section 3]). Hence, we have the following (identifying as usual the cohomology classes with the corresponding units via the standard Kummer map).

Proposition 2.1. *The element* $\kappa'(\chi,1) \in (\mathcal{O}_H[1/p]^\times \otimes L_\chi)^\chi$ *satisfies that*

$$L_p(\chi,1) = -\frac{1-p^{-1}}{\mathfrak{g}(\bar\chi)} \cdot \mathrm{ord}_p(\kappa'(\chi,1)).$$

Proof. This follows after combining the results of [Buy1, Section 6.2] on the properties of the map $\tilde{\mathcal{L}}_\chi$ with Solomon's computations, showing that the p-adic valuation of the derived class (sometimes referred as the *wild* cyclotomic unit) agrees with the logarithm of the circular unit (see also Proposition 4.1 of *loc. cit.*). □

3. Beilinson–Flach elements

3.1. **The three variable cohomology classes.** Let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ and $\mathbf{g}^* \in \Lambda_{\mathbf{g}}[[q]]$ be two Hida families of tame conductor N and tame nebentypus χ and $\bar\chi$, where $\Lambda_{\mathbf{g}}$ is a finite flat extension of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Let $\Lambda_{\mathbf{gg}^*} = \Lambda_{\mathbf{g}}\hat\otimes\Lambda_{\mathbf{g}}\hat\otimes\Lambda$, $\mathcal{W}_{\mathbf{gg}^*} = \mathrm{Spf}(\Lambda_{\mathbf{gg}^*})$ and consider also the $\Lambda_{\mathbf{g}}$-modules attached to the Hida families $\mathbf{g}$ and $\mathbf{g}^*$, that we denote by $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{g}^*}$, respectively. Finally, consider the $\Lambda_{\mathbf{gg}^*}$-module

$$\mathbb{V}_{\mathbf{gg}^*} := \mathbb{V}_{\mathbf{g}}\hat\otimes_{\mathbb{Z}_p}\mathbb{V}_{\mathbf{g}^*}\hat\otimes_{\mathbb{Z}_p}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}), \tag{8}$$

where $\Lambda(\underline{\varepsilon}_{\mathrm{cyc}}^{-1})$ stands for the twist of Λ by the inverse of the Λ-adic cyclotomic character. The formal spectrum of $\Lambda_{\mathbf{gg}^*}$ is endowed with certain distinguished points, the so-called crystalline points, denoted as $\mathcal{W}_{\mathbf{gg}^*}^\circ \subset \mathcal{W}_{\mathbf{gg}^*}$ and indexed by triples (y,z,σ); we refer the reader to Section 2 of *loc. cit.* for the definitions.

The Λ-adic Galois representation $\mathbb{V}_{\mathbf{gg}^*}$ is characterized by the property that for $(y,z,\sigma) \in \mathcal{W}_{\mathbf{gg}^*}^\circ$, with σ of weight $s \in \mathbb{Z}$, (8) specializes to

$$\mathbb{V}_{\mathbf{gg}^*}(y,z,\sigma) = V_{g_y} \otimes V_{g_z^*}(1-s),$$

the $(1-s)$-th Tate twist of the tensor product of the Galois representations attached to g_y and g_z^*.

Fix $c \in \mathbb{Z}_{>1}$ such that $(c, 6pN) = 1$. Then, [KLZ1, Theorem A] yields a three-variable Λ-adic global Galois cohomology class

$$\kappa^c(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*})$$

that is referred to as the Euler system of Beilinson–Flach elements associated with $\mathbf{g}$ and $\mathbf{g}^*$. We denote by $\kappa_p^c(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*})$ the image of $\kappa^c(\mathbf{g},\mathbf{g}^*)$ under the restriction map. Since c is fixed throughout, we may sometimes drop it from the notation. This constant does make an appearance in fudge factors accounting for the interpolation properties satisfied by the Euler system, but in all cases we are interested in these fudge factors do not vanish and hence do not pose any problem for our purposes.

Given a crystalline arithmetic point $(y,z,s) \in \mathcal{W}_{\mathbf{gg}^*}^\circ$ of weights (ℓ, m, s), set for notational simplicity throughout this section $g = g_y^\circ$, $g^* = (g_z^*)^\circ$. With these notations, g_y (resp. g_z^*) is the p-stabilization of g (resp. g^*) with U_p-eigenvalue α_g (resp. α_{g^*}).

Define

$$\kappa(g_y, g_z^*, s) := \kappa(\mathbf{g},\mathbf{g}^*)(y,z,s) \in H^1(\mathbb{Q}, V_{g_y} \otimes V_{g_z^*}(1-s)) \tag{9}$$

as the specialisation of $\kappa(\mathbf{g},\mathbf{g}^*)$ at (y,z,s).

As explained in [DR2, Section 2], the spaces $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{g}^*}$, as $G_{\mathbb{Q}_p}$-modules, are endowed with a stable filtration

$$0 \longrightarrow \mathbb{V}_{\mathbf{g}}^+ \longrightarrow \mathbb{V}_{\mathbf{g}} \longrightarrow \mathbb{V}_{\mathbf{g}}^- \longrightarrow 0,$$

where $\mathbb{V}_{\mathbf{g}}^{+}$ and $\mathbb{V}_{\mathbf{g}}^{-}$ are flat $\Lambda_{\mathbf{g}}$-modules with a $G_{\mathbb{Q}_p}$-action, locally free of rank one over $\Lambda_{\mathbf{g}}$, and such that the quotient $\mathbb{V}_{\mathbf{g}}^{-}$ is unramified. Define the $G_{\mathbb{Q}_p}$-subquotient $\mathbb{V}_{\mathbf{gg}^*}^{-+} := \mathbb{V}_{\mathbf{g}}^{-} \hat{\otimes} \mathbb{V}_{\mathbf{g}^*}^{+}$ of $\mathbb{V}_{\mathbf{gg}^*}$, which is of rank one over the two-variable Iwasawa algebra $\Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{g}^*}$ (this quotient makes sense because of [KLZ1, Proposition 8.1.7]).

Let $L_p(\mathbf{g}, \mathbf{g}^*)$ be the three-variable p-adic L-function characterized by the interpolation property of [Das, Theorem 3.7]. Although we will not need here an exact description of the different factors involved in its definition, let us note that the interpolation region is not symmetric on the weight ℓ and m, and in particular it satisfies $\ell > m$. The family of higher weight along the interpolation region is called *dominant*. Further, Theorem 2 of *loc. cit.* asserts that this function agrees with the two-variable p-adic L-function of the symmetric square, up to multiplication by a shifted p-adic zeta function.

The p-adic L-function $L_p(\mathbf{g}, \mathbf{g}^*)$ is intimately related with the class $\kappa(\mathbf{g}, \mathbf{g}^*)$. Indeed, one may consider the Perrin-Riou map

$$\langle \mathcal{L}_{\mathbf{gg}^*}^{-+}, \eta_{\mathbf{g}} \otimes \omega_{\mathbf{g}^*} \rangle : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*}^{-+} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \longrightarrow \Lambda_{\mathbf{gg}^*} \otimes \mathbb{Q}_p(\mu_N). \tag{10}$$

This application satisfies an explicit reciprocity law, which is the content of [KLZ1, Theorem B], and which asserts that

$$\langle \mathcal{L}_{\mathbf{gg}^*}^{-+}(\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)), \eta_{\mathbf{g}} \otimes \omega_{\mathbf{g}^*} \rangle = \mathcal{A}(\mathbf{g}, \mathbf{g}^*) \cdot L_p(\mathbf{g}, \mathbf{g}^*), \tag{11}$$

where $\mathcal{A}(\mathbf{g}, \mathbf{g}^*)$ is the Iwasawa function of [KLZ1, Theorem 10.2.2] and $\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)$ stands for the composition of the localization-at-p map with the projection $\mathbb{V}_{\mathbf{gg}^*} \to \mathbb{V}_{\mathbf{gg}^*}^{-+}$ in local cohomology.

The different specializations of the map $\langle \mathcal{L}_{\mathbf{gg}^*}^{-+}, \eta_{\mathbf{g}} \otimes \omega_{\mathbf{g}^*} \rangle$ can be expressed in terms of the Bloch–Kato logarithm or the dual exponential map. In particular, we are interested in the specializations of the class $\kappa(\mathbf{g}, \mathbf{g}^*)$ at weights $(1,1,0)$, and more generally, weights $(\ell, \ell, \ell-1)$, where the Perrin-Riou map interpolates, up to some explicit Euler factors, the Bloch–Kato logarithm. Unfortunately, these factors may vanish in the self-dual case, and one must resort to the concept of derivatives of Euler systems. In these cases, and to simplify the exposition, we shrink the weight space fixing a congruence class for ℓ modulo $p-1$ (that is, we restrict to $\ell \equiv 1 \pmod{p-1}$).

The following result is our starting point.

Proposition 3.1. *Let $\mathcal{C}$ denote the codimension two subvariety corresponding to the Zariski closure of the points $(\ell, \ell, \ell-1)$, that is,*

$$\mathcal{C} := \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{gg}^*} : y = z, \quad \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\}.$$

Then, $\kappa(\mathbf{g}, \mathbf{h})|_{\mathcal{C}} = 0$.

Proof. For points of the form $(\ell, \ell, \ell-1)$, with $\ell \geq 2$, this is a consequence of the vanishing of the Euler factor in [KLZ1, Theorem 8.1.3]. Then, the result follows by a limit argument using that the corresponding global cohomology module is free, as discussed e.g. in [BSV, Section 9.3]. □

Remark 3.2. Towards our further applications to the specializations at weight one, observe that with the notations presented in the Introduction $H^0(\mathbb{Q}, V_{gg^*}(1)) = 0$. This follows, for instance, noting that the module $V_{gg^*}(1)$ does not have any $G_{\mathbb{Q}_p}$-invariant (and therefore does not have $G_{\mathbb{Q}}$-invariants neither) by an analysis of the corresponding Hodge–Tate weights. More precisely, the weights of the representation along the line $(\ell, \ell, \ell-1)$ are $\{1, \ell, \ell, 2\ell-1\}$, which are never zero given our choice of the congruence class modulo $p-1$. In particular, this explains why the Λ-adic modules we will consider are torsion.

Remark 3.3. Proceeding exactly as in [BSV, Section 9.3], we may construct a (global) improved class $\kappa(\mathbf{g}, \mathbf{g}^*)^*$ along the surface corresponding to the closure of the points of weight

(ℓ, m, s), with $m = s-1$. Then, $\kappa(\mathbf{g}, \mathbf{g}^*) = \mathcal{E}_g(\mathbf{g}, \mathbf{g}^*) \cdot \kappa(\mathbf{g}, \mathbf{g}^*)^*$, where $\mathcal{E}_g(\mathbf{g}, \mathbf{g}^*)$ is an Euler factor which vanishes all along the line $\mathcal{C}$.

3.2. **Derivatives of Beilinson–Flach elements.** We keep the notations fixed in the introduction regarding weight one modular forms and units for the adjoint representation. Further, we fix a point of weight one $y_0 \in \mathrm{Spf}(\Lambda_{\mathbf{g}})$ such that $\mathbf{g}_{y_0} = g_\alpha$ and $\mathbf{g}^*_{y_0} = g^*_{1/\beta}$. As an extra piece of notation, let $\alpha'_{\mathbf{g}}$ stand for the derivative of the U_p-eigenvalue along the weight direction, and we just write α'_g for its evaluation at a certain specialization g of the Hida family.

In [RiRo1, Section 3] we constructed a derivative along the y-direction (alternatively, keeping y fixed and varying at a time the other two variables). For defining it, we had considered the curve

$$\mathcal{D} := \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{g}\mathbf{g}^*} : y = y_0, \quad \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\},$$

and showed that there exists a unique global class $\frac{\partial \kappa_p(\mathbf{g}, \mathbf{g}^*)}{\partial y} \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{g}\mathbf{g}^*|\mathcal{D}})$ such that

$$\kappa(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{D}} = \frac{\gamma - 1}{\log_p(\gamma)} \cdot \frac{\partial \kappa_p(\mathbf{g}, \mathbf{g}^*)}{\partial y},$$

where γ is a fixed topological generator. Further, we had established a reciprocity law (Theorem 3.10 in *loc. cit.*) expressing a suitable logarithm of this class in terms of special values of a three-variable p-adic L-function.

Note however that since the weight space is three-dimensional, it makes sense to ask about the derivative along any other direction (which are defined in a completely analogous way). Since along the line $(\ell, \ell, \ell-1)$ the class is identically zero, the derivative also vanishes. Hence, by an elementary argument in linear algebra, it suffices to determine the derivative along any other two *independent directions* to capture all the *first-order* information about the behavior of the class.

For the following Proposition we keep the notations discussed in the Introduction regarding the fields H and L, and also the unit u and the p-unit v. We consider the projection of the class to the adjoint component, and we still denote it with the same name: the reason is that one can show that the projection to the trivial component vanishes using that for odd twists [LZ, Cor. 4.1.3] there is a symmetry in the variables, and that the sum of the derivatives along both weight directions must lie in the adjoint piece.

To avoid annoying remarks after each of the results, we assume for the moment that

$$\log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \log_p(v_1) - \log_p(v_{\alpha\otimes\bar{\beta}}) \cdot \log_p(u_1) \neq 0,$$

and we will discuss at the end of the proof the degenerate case.

Lemma 3.4. *The derivative of* $\kappa_p(\mathbf{g}, \mathbf{g}^*)$ *at* $(y_0, y_0, 0)$ *along the* y*-direction (keeping fixed both* z *and* s*) satisfies the following equality in* $H^1(\mathbb{Q}_p, \mathrm{ad}^0(g)(1))$

$$\frac{\partial \kappa_p(\mathbf{g}, \mathbf{g}^*)}{\partial y}\Big|_{(y_0, y_0, 0)} = \Omega \cdot (\log_p(v_{\alpha\otimes\bar{\beta}})u - \log_p(u_{\alpha\otimes\bar{\beta}})v), \tag{12}$$

where $\Omega \in H_p$ *and we have made use of the usual notations for writing directional derivatives.*

Proof. According to the properties of the cohomology classes discussed in [RiRo1, Section 3.4] (see also [RiRo2, Section 4]) the left hand side may be written as a combination of the units u and v, which are a basis of the space $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(g)(1))$.

Then, the result follows by applying [KLZ1, Proposition 8.1.7] to $\kappa(\mathbf{g}, \mathbf{g}^*)$ in order to conclude that its projection to $\mathbb{V}^{--}_{\mathbf{g}\mathbf{g}^*}$ is identically zero, and therefore the same is true for its derivative. Specializing at $(y_0, y_0, 0)$, the result automatically follows. □

Remark 3.5. Observe that we will use the results of [LZ] which assert that the Beilinson–Flach elements lie in the part corresponding to the adjoint in the decomposition $V_{gg^*} = \mathrm{ad}^0(V_g) \oplus 1$

(so the same holds for any derivative which is symmetric in the weights). Alternatively, and following the discussion of [RiRo2, Section 4], one has that the projection of the subspace $p^{\mathbb{Z}}$ to the adjoint component is trivial.

We can now obtain an expression for $L_p^{g_\alpha}(g,g^*,0)$ involving the p-adic period Ω. In particular, considering the derivative along the analytic direction $(1,0,0)$ we have the following.

Proposition 3.6. *Up to multiplication by an element in $L^\times$, the following equality holds*

$$L_p^{g_\alpha}(g,g^*,0)\cdot\Big(\frac{-\alpha_{\mathbf{g}}'}{\alpha_g}\Big)_{|y_0} = \Omega\cdot(\log_p(u_{\alpha\otimes\bar\beta})\cdot\log_p(v_1)-\log_p(v_{\alpha\otimes\bar\beta})\cdot\log_p(u_1)).$$

Proof. This follows from making explicit the Euler factors in the explicit reciprocity law of [KLZ1, §10] and taking derivatives along the y-direction. □

We now obtain an analogous result for the derivative along the z-direction.

Lemma 3.7. *The derivative of $\kappa_p(\mathbf{g},\mathbf{g}^*)$ at $(y_0,y_0,0)$ along the z-direction (keeping fixed both y and s) satisfies the following equality in $H^1(\mathbb{Q}_p,\mathrm{ad}^0(g)(1))$ up to a factor in $L^\times$*

$$\frac{\partial\kappa_p(\mathbf{g},\mathbf{g}^*)}{\partial z}\Big|_{(y_0,y_0,0)} = \Omega\cdot(\log_p(v_{\alpha\otimes\bar\beta})u-\log_p(u_{\alpha\otimes\bar\beta})v). \tag{13}$$

Proof. We begin by noting that $L_p^{g_\alpha}(g_\alpha,g_{1/\beta}^*,0) = L_p^{g_{1/\beta}^*}(g_{1/\beta}^*,g_\alpha,0) \pmod{L^\times}$, where the latter corresponds to the function obtained by interpolating along the region where the Hida family attached to $g_{1/\beta}^*$ is dominant. This is immediate by Dasgupta's theorem [Das, Theorem 2], since the two factors arising in the decomposition does not depend on choosing either g_α or $g_{1/\beta}^*$ as the dominant factor (alternatively, it follows after a direct computation with Hida's factorization, see Proposition 4.3).

Moreover, for $h_{1/\alpha}=g_{1/\beta}^*$ we do have a relation between the derivatives of the U_p-eigenvalues, since

$$\frac{(1/\alpha_g)'}{1/\alpha_g} = -\frac{\alpha_g'}{\alpha_g}.$$

Proceeding as before, we may write now

$$\frac{\partial\kappa_p(\mathbf{g},\mathbf{g}^*)}{\partial z}\Big|_{(y_0,y_0,0)} = \Omega'\cdot(\log_p(v_{\alpha\otimes\bar\beta})u-\log_p(u_{\alpha\otimes\bar\beta})v).$$

Applying now the reciprocity law presented in [RiRo1, Theorem 3.7] and proceeding as in Proposition 3.6, we may conclude that $\Omega=\Omega' \pmod{L^\times}$. This follows since both the p-adic L-values and the logarithmic derivatives in the left hand side agree, as we have just discussed. (Note that the product of the periods arising when pairing with the differentials is a rational quantity, as discussed in [RiRo1, Section 5.2], so it does not affect the result.) □

Therefore, we may determine the derivative along the cyclotomic direction (keeping the weights fixed) by a linear algebra argument, and it is clear now that there is no contribution from the trivial component.

Proposition 3.8. *Assume that the derivative of $\kappa_p(\mathbf{g},\mathbf{g}^*)$ along the cyclotomic derivative is non-vanishing. Then, up to multiplication by scalar, the cyclotomic derivative of the Beilinson–Flach class $\kappa(\mathbf{g},\mathbf{g}^*)$ at $(y_0,y_0,0)$ is*

$$\frac{\partial\kappa_p(\mathbf{g},\mathbf{g}^*)}{\partial s}\Big|_{(y_0,y_0,0)} = \Omega\cdot(\log_p(v_{\alpha\otimes\bar\beta})u-\log_p(u_{\alpha\otimes\bar\beta})v) \pmod{L^\times},$$

where $\Omega\in H_p$ is the period of equation (12) and the equality holds in $H^1(\mathbb{Q}_p,V_{gg^*}(1))$.

Proof. Recall that the class vanishes along the line $(\ell, \ell, \ell-1)$. Hence, the result follows from equations (12) and (13) combined with the fact that

$$(0,0,1) = (1,1,1) - (1,0,0) - (0,1,0).$$

Note that although the derivatives along the y and the z direction do not necessarily agree, they do up to multiplication by $L^\times$, so the sum is also a multiple of that quantity by a factor in L. □

Observe that the previous results show that the different derived classes, which are elements living in a two-dimensional space, span the same line. In the next section, our aim is determining the value of the period Ω appearing in Proposition 3.8, which would complete the proof of our main theorem.

4. Cyclotomic derivatives and proof of the main theorem

4.1. **Cyclotomic derivatives.** Along this section, we assume that g and g^* do not move along Hida families and we just consider the cyclotomic variation. As an abuse of notation, write $\kappa(g, g^*, s) := \kappa(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, s)$ to emphasize the dependence on s. The image of this class under the Perrin-Riou map recovers the p-adic L-function $L_p^{g_\alpha}(g, g^*, s)$, that is, for $s \neq 0$,

$$\langle \mathcal{L}_{gg^*}^{-+}(\kappa_p^{-+}(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle = L_p^{g_\alpha}(g, g^*, s) \pmod{L^\times}. \tag{14}$$

(We have omitted the c-factor in the left since it is a non-vanishing rational factor when $s \neq 0$). Although we have shown that $\kappa_p(g, g^*, 0)$ is zero, we do not expect that $L_p^{g_\alpha}(g, g^*, 0) = 0$ in general. This is the same situation we previously found in the setting of circular units: the Kubota–Leopoldt p-adic L-function of a non-trivial, even Dirichlet character $L_p(\chi, s)$ is seen as the image of a Λ-adic cohomology class $\kappa(\chi, s)$ under a Perrin-Riou map; unfortunately, it happens that $\kappa(\chi, 1) = 0$ when $\chi(p) = 1$ and an Euler factor also vanishes, so we cannot assert (and indeed it is false!) that $L_p(\chi, 1) = 0$.

Along this section, and since there is no possible confusion, we write $\kappa'(g, g^*, s)$ for the cyclotomic derivative. Define the improved Perrin-Riou map as

$$\langle \widetilde{\mathcal{L}_{gg^*}^{-+}}, \eta_g \otimes \omega_{g^*} \rangle = \frac{\gamma - 1}{\frac{1}{p}\log_p(\gamma)} \times \langle \mathcal{L}_{gg^*}^{-+}, \eta_g \otimes \omega_{g^*} \rangle \; : \; H^1(\mathbb{Q}_p, V_{gg^*}^{-+}(1-s)) \longrightarrow \Lambda. \tag{15}$$

Therefore, we have

$$\langle \widetilde{\mathcal{L}_{gg^*}^{-+}}(\kappa_p'^{-+}(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle = p \cdot L_p^{g_\alpha}(g, g^*, s). \tag{16}$$

Hence, the value of $\langle \mathcal{L}_{gg^*}^{-+}(\kappa_p^{-+}(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle$ agrees with

$$\langle \widetilde{\mathcal{L}_{gg^*}^{-+}}(\kappa_p'^{-+}(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle. \tag{17}$$

Let $\hat{H}_p^\times$ denote the p-adic completion of $H_p^\times$. For the following result, consider as usual the identification

$$H^1(\mathbb{Q}_p, \mathrm{ad}^0(g)(1)) \simeq \hat{H}_p^\times[\mathrm{ad}^0(g)] \otimes L_p, \tag{18}$$

and take the element $\kappa_p'(g, g^*, 0)$, which belongs to the latter space (and may be therefore identified with a local unit in $\hat{H}_p^\times$). The same study of [Buy1, Remark 6.5] works verbatim in our setting, where he argues that the improved Perrin-Riou map is a multiple of the order map applied to the derived class. However, we want to find out this explicit multiple (at least, up to multiplication by a rational constant). Compare for example this setting with the computations of [LVZ, Proposition 2.5.5] and the discussions of Section 3 of *loc. cit.*, showing that the improved exponential map they consider is indeed the order map (up to sign).

Proposition 4.1. *Identifying* $\kappa_p'^{-+}(g,g^*,0)$ *with an element in* $(\hat{H}_p^\times \otimes L)^{G_{\mathbb{Q}_p}}$*, one has*

$$L_p^{g_\alpha}(g,g^*,0) = \mathrm{ord}_p(\kappa_p'^{-+}(g,g^*,0)) \pmod{L^\times}.$$

Proof. We can rephrase the statement in terms of the well-known theory of Coleman's power series. Then, $\kappa_p^{-+}(g,g^*,s)$ may be seen as a compatible system of units varying over the cyclotomic p-tower, but whose bottom layer is trivial. Hence, we may use the properties of universal norms and Coleman maps, and invoke the results developed in the proof of [Ven, Proposition 3.6], and more precisely (via duality) equation (27).

Then, the p-adic L-value can be obtained applying to $\kappa_p'(g,g^*,0)$ the map

$$\mathrm{ord}^{-+} \,:\, H^1(\mathbb{Q}, V_{gg^*} \otimes L_p(1)) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, L_p(1)) \xrightarrow{\widetilde{\mathcal{L}_{gg^*,0}^{-+}}} \mathbb{Q}_p,$$

where arguing as in [RiRo1, Section 5.2], the map $\widetilde{\mathcal{L}_{gg^*,0}^{-+}}$ corresponds to the usual p-adic order map multiplied by the p-adic period $\Omega_{g_\alpha}\Xi_{g^*_{1/\beta}}$, which belongs to $L^\times$.

According to (17), we conclude that

$$L_p(g,g^*,0) = \mathrm{ord}_p(\kappa_p'^{-+}(g,g^*,0)) \pmod{L^\times},$$

as desired. □

Roughly speaking, the previous theorem says that the derivative of the logarithm is the order (which can be seen as the dual of the result which interprets the derivative of the dual exponential as a logarithm).

Corollary 4.2. *With the notations introduced along the previous section, and up to multiplication by* $L^\times$*,*

$$L_p^{g_\alpha}(g,g^*,0) = \Omega \cdot \log_p(u_{\alpha\otimes\bar{\beta}}).$$

Proof. This directly follows by combining Propositions 4.1 and 3.8. □

This can be connected again with the case of circular units, that is, $L_p(g,g^*,s)$ is also the order of the derivative of $\kappa(g,g^*,s)$ along the s-direction.

4.2. **Improved p-adic L-functions.**

Let

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) := \frac{-\alpha_{\mathbf{g}}'(y_0)}{\alpha_{\mathbf{g}}(y_0)}, \tag{19}$$

where recall $\alpha_{\mathbf{g}} = a_p(\mathbf{g}) \in \Lambda_{\mathbf{g}}$ is the Iwasawa function given by the eigenvalue of the Hecke operator U_p acting on $\mathbf{g}$, and $\alpha_{\mathbf{g}}'$ is its derivative.

We finish the proof with the same argument invoked in [RiRo1], involving Hida's improved p-adic L-function, which we discussed in detail as Proposition 2.5 of *loc. cit.*

Proposition 4.3. *For a crystalline classical point* $y_0 \in \mathcal{W}_{\mathbf{g}}^\circ$ *of weight* $\ell \geq 1$*, we have*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,\ell) = L_p'(\mathrm{ad}^0(g_{y_0}),\ell),$$

up to a non-zero rational constant. Then, Theorem 1.1 in the introduction holds.

Proposition 4.4. *Assume that the* $\mathcal{L}$*-invariant* $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$ *is non-zero. Then, it may be written as*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \frac{\log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \log_p(v_1) - \log_p(v_{\alpha\otimes\bar{\beta}}) \cdot \log_p(u_1)}{\log_p(u_{\alpha\otimes\bar{\beta}})} \pmod{L^\times}.$$

Proof. Combining Proposition 3.6 with Proposition 4.1, we have that

$$\frac{\Omega}{\mathcal{L}(\mathrm{ad}^0(g_\alpha))} \cdot \Big(\frac{\log_p(u_{\alpha\otimes\bar\beta}) \cdot \log_p(v_1) - \log_p(v_{\alpha\otimes\bar\beta}) \cdot \log_p(u_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \Big) = \Omega \pmod{L^\times}.$$

Dividing by Ω (provided that this value is non zero), the result follows.

Note that if Ω were zero, by 4.2, we know that $L_p(g,g^*,0)=0$, too. But this value with the $\mathcal{L}$-invariant, that we have assumed that it is non-zero. ☐

Remark 4.5. If the $\mathcal{L}$-invariant is zero, we have that

$$L_p(g,g^*,0) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \Omega = 0,$$

which is a situation we cannot rule out. In this case, indeed, any directional derivative of the Beilinson–Flach class is zero and our study is meaningless since both the p-adic L-function and the Euler system vanish.

5. A REINTERPRETATION OF THE SPECIAL VALUE FORMULA

The results discussed along this article were presented in the introduction as a special value formula for the Hida–Rankin p-adic L-function. Alternatively, they can be regarded as the computation of the $\mathcal{L}$-invariant for the adjoint of a weight one modular form, and following the original formulation given by Darmon, Lauder, and Rotger, it also admits a reinterpretation in terms of a p-adic iterated integral. This point of view is specially useful towards computational experiments following Lauder's algorithms [La], as it is further discussed in [DLR, Section 3] and [DLR2, Section 1]. It may be instructive for the reader to look at the discussions in loc. cit., since they offer a broad picture of the algorithmic side of the story, which was crucial for the formulation of the conjectures and for having a better understanding of the results.

In order to give a more conceptual view of our results, and how they fit in the theory of exceptional zeros and Galois deformations of modular forms, we would like two discuss two other interpretations which were already behind the scenes in our joint works with Rotger. This section may be safely skipped, and we have included it here to illustrate an alternative interpretation of our results in the setting of deformation theory.

5.1. **Deformations of weight one modular forms.** As usual, fix a p-stabilization g_α of the weight one modular form $g \in S_1(N,\chi)$. We discuss a reinterpretation of the main results in terms of deformations of modular forms, in a striking analogy with the different works around the Gross–Stark conjecture, and which may be useful towards generalizations of the main results to totally real fields, following the recent approach of Dasgupta, Kakde, and Ventullo [DKV].

Let E_k denote the weight k Eisenstein series, whose Fourier expansion is given by

$$E_k = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad \text{where } \sigma_{k-1}(n) = \sum_{d\,|\,n} d^{k-1}. \tag{20}$$

There are two possible ways of considering its p-adic variation in families, either by taking the ordinary p-stabilization, E_k^{ord}, or the critical one, E_k^{crit}. For the sake of simplicity, we restrict to the ordinary p-stabilization, and after further normalizing by $\zeta(1-k)/2$, we have the usual Eisenstein series $G_k^{(p)}$, given by

$$G_k^{(p)} = 1 + 2\zeta_p(1-k)^{-1} \sum_{n=1}^{\infty} \sigma_{k-1}^{(p)}(n) q^n.$$

Since $\zeta_p(1-k)$ has a pole at $k=0$, the previous expression is not defined at that point, although we may formally interpret it as $G_0^{(p)} = 1$. It makes sense to consider its derivative (at least formally), and write

$$(G_0^{(p)})' := 2(1-p^{-1})^{-1} \cdot \sum_{n=1}^{\infty} \sigma_{-1}^{(p)} q^n.$$

Further, we may take the infinitesimal deformation $G_0^{(p)} + \varepsilon (G_0^{(p)})'$, and then multiplying by g_α we obtain

$$(G_0^{(p)} + \varepsilon (G_0^{(p)})') \cdot g_\alpha = g_\alpha + \varepsilon (G_0^{(p)})' g_\alpha. \tag{21}$$

We regard this expression as a modular form of weight $1+\varepsilon$ corresponding to an infinitesimal deformation of g_α.

There is another natural deformation of g_α we want to consider, which is precisely the one behind the scenes in [RiRo1] and which also appeared in [DLR2]. This is defined as

$$g_\alpha' := \Big(\frac{d}{dy} \mathbf{g}_\alpha \Big)_{|y=y_0},$$

which is a generalized eigenform in the same (generalised) eigenspace. Then, we may take a second deformation of the modular form g_α, given by

$$g_\alpha + \varepsilon g_\alpha'. \tag{22}$$

Let e_{ord} stand for the ordinary projector, and e_{g_α} for the projector onto the g_α-isotypic component.

Proposition 5.1. *Under the running assumptions,*

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha G_0^{[p]}) = (1-\alpha_g U_p^{-1}) g_\alpha' \pmod{L^\times}.$$

Proof. Substracting the deformations in (21) and (22), we obtain $g_\alpha (G_0^{(p)})' - g_\alpha'$. If we furthermore take the ordinary projection and project to the g_α-component, we obtain a multiple of g_α, that is

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha (G_0^{(p)})') = g_\alpha' + \mathcal{L} \cdot g_\alpha. \tag{23}$$

Next, if we apply the operator $1-\alpha_g U_p^{-1}$ to both sides of the previous equation, the left hand side becomes just the p-depletion

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha (G_0^{[p]})'), \tag{24}$$

while in the right hand side the operator $1-\alpha_g U_p^{-1}$ annihilates g_α. We have thus proved the result. □

Note that the left hand side is an explicit multiple of the p-adic L-function $L_p^{g_\alpha}(g, g^*, 0)$, so the proposition asserts that the $\mathcal{L}$-invariant which governs the arithmetic of the adjoint may be read as a generalized eigenvalue attached to the deformation g_α', that is,

$$(1-\alpha_g U_p^{-1}) g_\alpha' = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot g_\alpha \pmod{L^\times}. \tag{25}$$

5.2. **Spaces of generalized eigenvectors.** This section is merely speculative and tries to shed some light into very natural questions around the previously discussed phenomena. When discussing circular units, we have seen that the condition $\chi(p)=1$ provides us with a p-unit in an extended Selmer group, and we have discussed how to mimic this approach in the case of Beilinson–Flach elements. But more generally, given two modular forms g and h of weights ℓ and m, respectively, and an integer s, there is a geometric construction of the so-called Eisenstein classes $\mathrm{Eis}^{[g,h,s]}$ whenever the triple (ℓ, m, s) satisfies the *weight condition* of [KLZ1, Section 7], that is,

$$1 \leq s < \min\{\ell, m\}.$$

This includes all the points of weights $(\ell, \ell, \ell-1)$ when $\ell \geq 2$, and in particular the classical weight two situation. This suggests an interpretation of the derived class as a limit of Eisenstein classes (which are generically non-vanishing) for weights $\ell \geq 2$.

The proof of the explicit reciprocity law for Beilinson–Flach classes rests on an explicit connection between the p-depleted Eisenstein series (which encodes values of the p-adic L-function) and the p-stabilized one (which encodes values of the regulator of a geometric cycle). In this note we recovered the expressions for the logarithm of the derived class in terms of p-adic L-values, but it is natural to look for a reciprocity law involving $\mathrm{Eis}^{[g,h,s]}$, whenever $h = g^*$ and $s = \ell - 1$. Note that this was the only case excluded by [KLZ2, Theorem 6.5.9]. Let us discuss the limitations for a result like that and that one may find natural in this framework.

According to [DR1, Lemma 4.10] and [KLZ2, Lemma 6.5.8], the quantity $g \times h^{[p]}$ can be expressed in terms of $g_\alpha h^{(p)}$ using the operator $1 - \alpha_g \cdot U_p^{-1}$, where $h^{[p]}$ (resp. $h^{(p)}$) stands for the p-depletion (resp. p-stabilization) of the modular form h. In the non self-dual case, the corresponding operator acting on the space of generalized eigenforms $S_1(Np)[[g_\alpha]]$ is invertible, and we obtain a straightforward linear relation which is crucially used in *loc. cit.*

However, when $h = g^*$, the connection is more involved. In this case, consider a generalized eigenbasis $\{e_1, \ldots, e_n\}$ for the U_p-operator acting on the space of generalized (non-necessarily overconvergent) modular forms $S_1(Np)[[g_\alpha]]$, that is,

$$U_p \cdot e_1 = \alpha_g \cdot e_1, \quad U_p \cdot e_2 = e_1 + \alpha_g \cdot e_2, \quad \ldots, \quad U_p \cdot e_n = e_{n-1} + \alpha_g \cdot e_n. \tag{26}$$

Hence, the matrix corresponding to the operator $1 - \alpha_g \cdot U_p^{-1}$ acting on this space has the quantity $-1/\alpha_g$ all over the upper diagonal and zeros elsewhere. If we now apply this operator to $E_0^{(p)} g_\alpha$, written in this basis as $\sum \lambda_i e_i$, what we get in the first non-zero component is $-1/\alpha_g \cdot \lambda_2$. That is, the second vector of the generalized eigenbasis is the one which encodes the p-adic L-value. Therefore, one may consider two different classes.

(a) The class $\mathrm{Eis}^{[g,g^*,\ell-1]}$, where ℓ is the weight of g, is related with the first coefficient in the expansion in the generalized eigenbasis (see [KLZ1, Corollary 6.5.7]). This controls the p-stabilization of the Eisenstein series.

(b) The derived class $\kappa'(g, g^*)$, constructed in [RiRo1], is related with the p-adic L-value, and hence with second coefficient in the generalized eigenbasis. This measures the p-depletion of the Eisenstein series.

Hence, when $g \in S_\ell(N, \chi_g)$ is an ordinary modular form of weight $\ell \geq 2$, the two classes

$$\{\mathrm{Eis}^{[g,g^*,\ell-1]}, \kappa'(g, g^*)\}$$

are a priori unrelated.

Question 5.2. Can we interpret the class $\mathrm{Eis}^{[g,g^*,\ell-1]}$ in terms of some p-adic L-value? And can we make sense of the limit of these classes for weights $(\ell, \ell, \ell-1)$ when ℓ goes to 1 p-adically?

Note that while the p-depleted class is connected with the usual p-adic L-value, a priori there is no natural p-adic avatar encoding the value of the p-stabilized class.

Observe that in the setting of diagonal cycles of [BSV], the authors take a different approach to the vanishing phenomenon, defining an *improved* class which is a putative geometric refinement to the analogue of the Eisenstein class, and which agrees up to some $\mathcal{L}$-invariant with the derived class.

6. Towards a p-adic Harris–Venkatesh conjecture

It is a somewhat vexing fact that our computations regarding the $\mathcal{L}$-invariant of the adjoint of a weight one modular form captures a 2×2 regulator encoding information about both a unit and a p-unit, while the most natural object to work would be the unit itself, as it occurs with the celebrated Gross–Stark conjecture. Similarly, one would expect to be able to construct an *Eisenstein p-adic L-function*, in such a way that appropriate special values of it also capture information about the Beilinson–Flach classes, in a way that we now make precise. This section is purely conjectural, and must be regarded as a failure in our current work, where we have not succeeded in studying these aspects.

6.1. **Motivic p-adic L-functions.** Consider the most general setting in which g and h are two weight one modular forms. As we have already recalled, there are four Beilinson–Flach classes attached to the choice of p-stabilizations of g and h,

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta). \tag{27}$$

We know that different components of it are related to special values of p-adic L-functions. Take for instance the case of $\kappa(g_\alpha, h_\alpha)$. Considering its restriction to a decomposition group at p, and under the assumption that $\alpha \neq \beta$, we may take as in [DR2, Section 2] a decomposition of $\kappa_p(g_\alpha, h_\alpha)$ of the form

$$\kappa_p^{--}(g_\alpha, h_\alpha) \otimes e_{\beta\beta}^\vee \oplus \kappa_p^{-+}(g_\alpha, h_\alpha) \otimes e_{\beta\alpha}^\vee \oplus \kappa_p^{+-}(g_\alpha, h_\alpha) \otimes e_{\alpha\beta}^\vee \oplus \kappa_p^{++}(g_\alpha, h_\alpha) \otimes e_{\alpha\alpha}^\vee, \tag{28}$$

where $\{e_{\alpha\alpha}^\vee, e_{\alpha\beta}^\vee, e_{\beta\alpha}^\vee, e_{\beta\beta}^\vee\}$ is a basis for $V_{gh}^\vee = \mathrm{Hom}(V_{gh}, L)$, where

$$\mathrm{Fr}_p(e_{\alpha\alpha}^\vee) = \chi_{gh}^{-1}(p)\beta_g\beta_h \cdot e_{\alpha\alpha}^\vee, \ \ldots, \ \mathrm{Fr}_p(e_{\beta\beta}^\vee) = \chi_{gh}^{-1}(p)\alpha_g\alpha_h \cdot e_{\beta\beta}^\vee.$$

According to [KLZ1, Proposition 8.2.6], the component $\kappa_p^{--}(g_\alpha, h_\alpha) = 0$ vanishes. In the same way, the components $\kappa_p^{-+}(g_\alpha, h_\alpha)$ and $\kappa_p^{+-}(g_\alpha, h_\alpha)$ are related to the special values of the Hida–Rankin p-adic L-functions $\mathscr{L}_p{}^{g_\alpha}$ and $\mathscr{L}_p{}^{h_\alpha}$, respectively. Hence, it is natural to expect that the remaining component $\kappa_p^{++}(g_\alpha, h_\alpha)$ could arise as the special value of some p-adic L-function.

Following the analogy with the case of diagonal cycles and triple product p-adic L-functions, it would be attached to the triple $(E_2(1, \chi_{gh}^{-1}), g, h)$, but varying over the region where the Eisenstein family is dominant. Of course this is not possible (at least without any further modification), but let us work formally in terms of the theory of Perrin-Riou maps. In particular, we may consider the three-variable cohomology class $\kappa(\mathbf{g}, \mathbf{h})$, take the restriction to the line where both g and h are fixed and take the image under the Perrin-Riou map. That way we would get an element over the Iwasawa algebra that we may denote $L_p^{\mathrm{Eis}}(g, h, s)$. It may be instructive to compare this with the scenario of triple products, where the existence of a third p-adic L-function $\mathscr{L}_p{}^f$, which at points of weight $(2,1,1)$ interpolates classical L-values, provides a richer framework and draws a more complete picture.

To simplify things and discuss these phenomena in the framework of the note, let us focus just on the case where both g and h are self dual, that is $h = g^*$, and keep the assumptions (H1)-(H3). Recall that this situation naturally splits in two scenarios, namely $h_\alpha = g_{1/\beta}^*$ and $h_\alpha = g_{1/\alpha}^*$. As we have mentioned before, we expect the previous *cyclotomic* p-adic L-function to encode information about the logarithm of the unit u, and not about the apparently complicated regulator of our main result.

More concretely, assume that $\alpha_g\alpha_h = 1$, and take the class $\kappa(g_\alpha, g_{1/\alpha}^*)$, although the same works verbatim for $\kappa(g_\beta, g_{1/\beta}^*)$. From the general theory of Perrin-Riou maps, we may consider the map

$$\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle \, : \, H^1(\mathbb{Q}_p, V_{gg^*}^{++}(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \longrightarrow I^{-1}\Lambda_{\mathbf{g}}, \tag{29}$$

with specializations

$$\nu_s(\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle) \, : \, H^1(\mathbb{Q}_p, V_{gg^*}^{++}(1-s)) \longrightarrow \mathbb{C}_p,$$

where

$$\nu_s(\langle \mathcal{L}^{++}_{gg^*}, \omega_g \otimes \omega_{g^*}\rangle) = \frac{1-p^{s-1}}{1-p^{-s}} \cdot \begin{cases} \langle \frac{(-1)^s}{(-s)!} \cdot \langle \log_{\mathrm{BK}}, \omega_g \otimes \omega_{g^*}\rangle & \text{if } s<0 \\ (s-1)! \cdot \langle \exp^*_{\mathrm{BK}}, \omega_g \otimes \omega_{g^*}\rangle & \text{if } s>1. \end{cases}$$

Observe that we have not said anything about the specializations at $s=0$ and at $s=1$.

When $s=0$ (resp. $s=1$), we are still in the region of the Bloch–Kato logarithm (resp. dual exponential map), but the expression $1-p^{-s}$ (resp. $1-p^{s-1}$) vanishes. Rescaling the classes appropriately as discussed later on, we may get the image of the previous map contained in $\Lambda_{\mathbf{g}}$. Hence, and following [LVZ, Proposition 2.5.5] (see also the computations of [Ven, Section 3.1] and [Buy1, Section 6.3]), we have the following.

Lemma 6.1. *The map*

$$\nu_0(\langle \mathcal{L}^{++}_{gg^*}, \omega_g \otimes \omega_{g^*}\rangle) \; : \; H^1(\mathbb{Q}_p, V^{++}_{gg^*}(1)) \longrightarrow \mathbb{C}_p \tag{30}$$

is given by

$$\nu_0(\langle \mathcal{L}^{++}_{gg^*}, \omega_g \otimes \omega_{g^*}\rangle) = (1-p^{-1}) \cdot \langle \mathrm{ord}_p, \omega_g \otimes \omega_{g^*}\rangle. \tag{31}$$

Remark 6.2. Compare this situation with the case of circular units: there, the fact of taking the derived class was related to the fact that the Coleman map was connected to an *imprimitive* p-adic L-function, vanishing at the point of interest and whose derivative there corresponds to the special value of the Kubota–Leopoldt p-adic L-function.

Define

$$L_p^{\mathrm{mot}}(g_\alpha, g^*_{1/\alpha}, s) = \langle \mathcal{L}^{++}_{gg^*}(\kappa_p^{++}(g_\alpha, g^*_{1/\alpha}, s)), \omega_g \otimes \omega_{g^*}\rangle, \tag{32}$$

where $\kappa(g_\alpha, g^*_{1/\alpha}, s)$ is the restriction of the 3-variable cohomology class to the cyclotomic line, followed by multiplication by $\frac{\gamma-1}{\frac{1}{p}\log_p(\gamma)}$.

As a piece of notation for the following result, let $\mathcal{L}_{g_\alpha}$ stand for the period ratio introduced in [DR2, Section 2].

Proposition 6.3. *The special value of the derivative of $L_p^{\mathrm{mot}}(g_\alpha, g^*_{1/\alpha}, 0)$ satisfies that*

$$L_p^{\mathrm{mot}}(g_\alpha, g^*_{1/\alpha}, 0) = \frac{\mathcal{L}_{g_\alpha}}{\log_p(u_{\alpha\otimes\bar\beta})} \times \log_p(u_1) \pmod{L^\times}.$$

Proof. When $s=0$, the denominator of the Perrin-Riou map $\mathcal{L}^{++}_{gg^*}$ vanishes and we are in the setting discussed before. Then, the Perrin-Riou map is given by the order followed by the pairing with the canonical differentials, as in (31). Since according to the results of [RiRo2, Section 4]

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \frac{1}{\Xi_{g_\alpha}\Omega_{g^*_{1/\alpha}}} \frac{\log_p(u_1)\cdot v - \log_p(v_1)\cdot u}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times},$$

the image of $\kappa_p(g_\alpha, g^*_{1/\alpha})$ under the map (31) agrees with

$$\log_p(u_1) \cdot \frac{\mathcal{L}_{g_\alpha}}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times}.$$

□

Further, recall that according to [DR2, Conjecture 2.3], we expect that $\mathcal{L}_{g_\alpha}$ must agree with $\log_p(u_{\alpha\otimes\bar\beta})$ up to multiplication by $L^\times$, and this would give just $\log_p(u_1)$ in the previous formula.

As a final comment, observe that the class $\kappa(g_\alpha, g^*_{1/\beta})$ vanishes, while neither the corresponding numerator nor the denominator of the Perrin-Riou map do. Hence, we expect the special value to be zero. However, it would be licit to take the derivative of both the class and the p-adic L-function.

6.2. **Critical Eisenstein series.** We close this article with a more speculative section, devoted to highlight the connection with the arithmetic of triple products. Let E_{2k} be the Eisenstein series of weight $2k$ and trivial characters, and consider its critical p-stabilization E_{2k}^{crit}. Then, there is a Coleman family $\mathbf{f}$ passing through E_{2k}^{crit}, and we may consider the triple product p-adic L-function attached to $(\mathbf{f}, g_\alpha, g_{1/\alpha}^*)$, where we fix the second and third modular forms (and do not move them along their families). Let x stand for the weight variable of $\mathbf{f}$ and x_0 for the point such that $\mathbf{f}_{x_0} = E_{2k}^{\mathrm{crit}}$. Then, one can easily check that the product of the p-adic adjoint L-function of $\mathbf{f}$ and $\mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, g_{1/\alpha}^*)$ at the critical Eisenstein specialization vanishes, since it can be recast as the projection of a cusp form to the Eisenstein component. However, this is due to the vansihing of the adjoint, so it is still natural to make the following definition.

Definition 6.4. For any positive integer $k \geq 2$, let

$$\delta_k = \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, g_{1/\alpha}^*)(x_0).$$

Note that the different values of δ_k are a priori unrelated, and they need not to be the values of any analytic function (the different Coleman critical Eisenstein series do not live in the same family).

The general Perrin-Riou formalism establishes a connection between p-adic L-functions and Euler systems, which in this case conjecturally links the value δ_k with a suitable class in $H^1(\mathbb{Q}_p, V_{gg^*}^{++}(k))$ (coming from the arithmetic of diagonal cycles). However, with the current methods we are unable to prove a result of this kind, since it would require a deeper understanding of the explicit reciprocity laws at critical Eisenstein points. The following conjecture seems thus a natural degeneration of the results of [BSV] to the critical Eisenstein scenario.

Conjecture 6.5. *There exists a rigid analytic function* $\mathfrak{f}(k)$ *such that*

$$\delta_k = \mathfrak{f}(k) \cdot L_p^{\mathrm{mot}}(g_\alpha, g_{1/\alpha}^*, 1-k) \pmod{L^\times} \tag{33}$$

In particular, $\mathfrak{f}(1)$ measures the relation between the logarithm of the unit of the adjoint, $\log_p(u_1)$, and an analytic avatar attached to the triple $(E_2, g_\alpha, g_{1/\alpha}^*)$, on the realm of the Harris–Venkatesh conjecture. This avatar may be understood (up to multiplication by the derivative of the p-adic adjoint L-value) as the p-adic limit when ϵ goes to zero of the (derivative of the) pairing between a cusp form of weight $2+\epsilon$ converging to E_2 and $d^{\epsilon/2}gg^*$, where here d is a suitable derivative operator.

A result of this kind is motivated by our forthcoming work with Loeffler on *Eisenstein critical* Euler systems, and also by the recent article of Benois and Büyükboduk [BeBu], which in a similar situation establish a similar connection between a *weight derivative* and a motivic p-adic L-function. Unfortunately, their work excludes the Eisenstein case, but we hope to explore this connection in further work.

Remark 6.6. In cases where an exceptional extra vanishing occurs, one would need to take higher derivatives of the triple product L-function and consider instead $\frac{\partial^r \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, g_{1/\alpha}^*)}{\partial x^r}$, where r is the order of vanishing at x_0.

References

[BeDi] J. Bellaiche, M. Dimitrov, *On the eigencurve at classical weight one points*, *Duke Math. J.* **165** (2016), 245–266.

[Ben1] D. Benois. *Trivial zeros of p-adic L-functions at near central points*, *J. Inst. Math. Jussieu* **13**, no. 3 (2014), 561–598.

[Ben2] D. Benois. *On extra-zeros and p-adic L-functions: the crystalline case*, *“Iwasawa theory 2012. State of the Art and Recent Advances”, Contributions in Mathematic and Computational Science* **7**, Springer (2015).

[BeBu] D. Benois and K. Büyükboduk. *Interpolation of Beilinson–Kato elements and p-adic L-functions*, *Ann. Math. Québec* **46** (2022), 231–287.

[BD] M. Bertolini, H. Darmon. *Hida families and rational points on elliptic curves*, *Inventiones mathematicae* **168** (2007), 371–431.

[BSV] M. Bertolini, M. Seveso, and R. Venerucci. *Reciprocity laws for balanced diagonal classes*, *Astérisque* **434** (2022), 175–201.

[BeDi] A. Betina and M. Dimitrov. *Geometry of the eigencurve at CM points and trivial zeros of Katz p-adic L-functions*, *Adv. Math.* **384** (2021).

[BDP] A. Betina, M. Dimitrov, and A. Pozzi, *On the Gross–Stark conjecture*, *Amer. J. Math.* **144** (2022), no. 1, 227–265.

[Buy1] K. Büyükboduk. *Height pairings, exceptional zeros and Rubin's formula*, *Comment. Math. Helv.* **87** (2012), no. 1, 71–111.

[Buy2] K. Büyükboduk. *On Nekovar's heights, exceptional zeros and a conjecture of Mazur-Tate-Teitelbaum Int. Math. Res. Not* (2016), 2197–2237.

[CW77] J. Coates and A. Wiles. *On the conjecture of Birch and Swinnerton-Dyer*, *Invent. Math.* **39** (1977), no. 3, 223–251.

[DDP] H. Darmon, S. Dasgupta and R. Pollack. *Hilbert Modular Forms and the Gross–Stark Conjecture, Ann. of Math* **174** (2011), no. 1, 439–484.

[DHRV] H. Darmon, M. Harris, V. Rotger, and A. Venkatesh. *Derived Hecke algebra for dihedral weight one forms*, *Michigan Math. J.* **72** (2022), 145–207.

[DKV] S. Dasgupta, M. Kakde, and K. Ventullo, *On the Gross-Stark conjecture*, *Annals of Math.* **188** (2018), no. 3, 833–870.

[DLR] H. Darmon, A. Lauder, and V. Rotger. *Gross-Stark units and p-adic iterated integrals attached to modular forms of weight one*, *Ann. Math. Québec*, volume dedicated to Prof. Glenn Stevens on his 60th birthday, **40** (2016), 325–354.

[DLR2] H. Darmon, A. Lauder, and V. Rotger. *First order p-adic deformations of weight one newforms*, in "L-functions and automorphic forms", *Contr. in Math. and Comp. Sc.* **12**.

[DR1] H. Darmon and V. Rotger. *Diagonal cycles and Euler systems I: a p-adic Gross-Zagier formula*, Annales Scientifiques de l'Ecole Normale Supérieure, **47** (2014), no. 4, 779–832.

[DR2] H. Darmon and V. Rotger. *Elliptic curves of rank two and generalised Kato classes*, *Research in Math. Sciences* **3:27** (2016).

[Das] S. Dasgupta, *Factorization of p-adic Rankin L-series*, *Invent. Math.* **205** (2016), no. 1, 221–268.

[FG] B. Ferrero and R. Greenberg, *On the Behavior of p-adic L-functions at $s = 0$*, *Invent. Math.* **50** (1978), no. 1, 91–102.

[Gr] R. Greenberg. *Trivial zeros of p-adic L-functions, p-adic monodromy and the Birch and Swinnerton-Dyer conjecture Contemp. Math*, (1991), 149–174.

[GS] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, *Invent. Math.* **111** (1993), no. 2, 407–447.

[Gross] B. Gross, *p-adic L-series at $s = 0$*, *Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981-82), no. 3, 979–994.

[GK] B. Gross and N. Koblitz, *Gauss Sums and the p-adic Γ-function*, *Ann. of Math.* **109** (1979), no. 3, 569–581.

[HV] M. Harris and A. Venkatesh. *Derived Hecke algebra for weight one forms*, *Exp. Math.* **28** (2019), no. 3, 342–361.

[Hi1] H. Hida, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms I*, *Invent. Math.* **79** (1985), no. 1, 159–195.

[Hi2] H. Hida, *A p-adic measure attached to the zeta functions associated with two elliptic modular forms II*, *Ann. Inst. Fourier (Grenoble)* **38** (1988), no. 3, 1–83.

[KLZ1] G. Kings, D. Loeffler, and S.L. Zerbes. *Rankin-Eisenstein classes and explicit reciprocity laws*, *Cambridge J. Math* **5** (2017), no. 1, 1–122.

[KLZ2] G. Kings, D. Loeffler, and S.L. Zerbes. *Rankin-Eisenstein classes and explicit reciprocity laws*, *Amer. J. Math* **142** (2020), no. 1, 79–138.

[Ko] S. Kobayashi, *An elementary proof of the Mazur–Tate–Teitelbaum conjecture for elliptic curves*, *Documenta Math.*, Extra Volume (2006), 567–575.

[La] A. Lauder, *Computations with classical and p-adic modular forms*, *LMS J. Comput. Math.* **14** (2011), 214–231.

[LVZ] D. Loeffler, O. Venjakob and S.L. Zerbes. *Local epsilon-isomorphisms*, *Kyoto J. Math* **55** (2015), no. 1, 63–127.

[LZ] D. Loeffler and S.L. Zerbes. *Iwasawa theory for the symmetric square of a modular form*, *J. Reine Angew. Math.* **752** (2019), 179–210.

[LZ2] D. Loeffler and S.L. Zerbes. *P-adic L-functions and diagonal cycles for GSp(4) x GL(2) x GL(2)*, preprint (2020).

[MTT] B. Mazur, J. Tate, and J. Teitelbaum. *On p-adic analogs of the conjectures of Birch and Swinertonn-Dyer*, *Invent. Math.* **84** (1986), 1-48.

[Ri] O. Rivero. *The exceptional zero phenomenon for elliptic units*, *Rev. Mat. Iberoam.* **37** (2021), no. 1333–1364.

[RiRo1] O. Rivero and V. Rotger. *Derived Beilinson–Flach elements and the arithmetic of the adjoint of a modular form*, *J. Eur. Math. Soc.* **23** (2021), no. 7, 2299–2335.

[RiRo2] O. Rivero and V. Rotger, *Beilinson-Flach elements and Stark conjectures*, *Forum Math.* **31** (2019), no. 6, 1517–1532.

[RRV] M. Roset, V. Rotger, and V. Vatsal. *On the L-invariant of the adjoint of a weight one modular forms*, *J. London Math. Soc.* **104** (2021), no. 1, 206–232.

[Ros] G. Rosso. *L-invariant for Siegel–Hilbert forms*, *Documenta Math.* **20** (2015), 1227–1253.

[Ven] R. Venerucci. *Exceptional zero formulae and a conjecture of Perrin-Riou*, *Invent. Math.* **203** (2016), 923-972.

O. R.: MATHEMATICS INSTITUTE, ZEEMAN BUILDING, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UK

Email address: `Oscar.Rivero-Salgado@warwick.ac.uk`