ON p-ADIC L-FUNCTIONS FOR $\mathrm{GSp}_4 \times \mathrm{GL}_2$

DAVID LOEFFLER AND ÓSCAR RIVERO

Abstract. We use higher Coleman theory to construct a new p-adic L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$. While previous works by the first author, Pilloni, Skinner and Zerbes had considered the p-adic variation of classes in the H^2 of Shimura varieties for GSp_4, in this note we explore the interpolation of classes in the H^1, which allows us to access to a different range of weights. Further, we show an interpolation property in terms of complex L-values using the algebraicity results established in previous work by the authors.

Contents

1. Introduction

Let π and σ be cuspidal automorphic representations of $\mathrm{GSp}_4 / \mathbf{Q}$ and $\mathrm{GL}_2 / \mathbf{Q}$ respectively. Then we have a degree 8 L-function $L(\pi \times \sigma, s)$, associated to the tensor product of the natural degree 4 (spin) and degree 2 (standard) representations of the L-groups of GSp_4 and GL_2. If π and σ are algebraic, then this L-function is expected to correspond to a motive, and in particular we can ask whether it has critical values.

We suppose that π (or, more precisely, its L-packet) corresponds to a holomorphic Siegel modular eigenform of weight (k_1, k_2), for $k_1 \geqslant k_2 \geqslant 2$ integers, and that σ corresponds to a holomorphic elliptic modular form of weight $\ell \geqslant 1$. For $L(\pi \times \sigma, s)$ to be a critical value, we must have $s = \frac{-(k_1+k_2+\ell-4)}{2} + j$ for $j \in \mathbf{Z}$, so that $L(\pi \times \sigma, s) = L(V_p(\pi) \otimes V_p(\sigma), j)$ where $V_p(-)$ are the Galois representations corresponding to π and σ; and the tuple (k_1, k_2, ℓ, j) has to satisfy one of 3 different sets of (mutually exclusive) inequalities, which we have outlined in more detail in the companion paper [LR23], corresponding to the cases (A), (D), (F) in Table 1 of *op.cit.*. In this paper, we focus on region (D), which is given by the inequalities

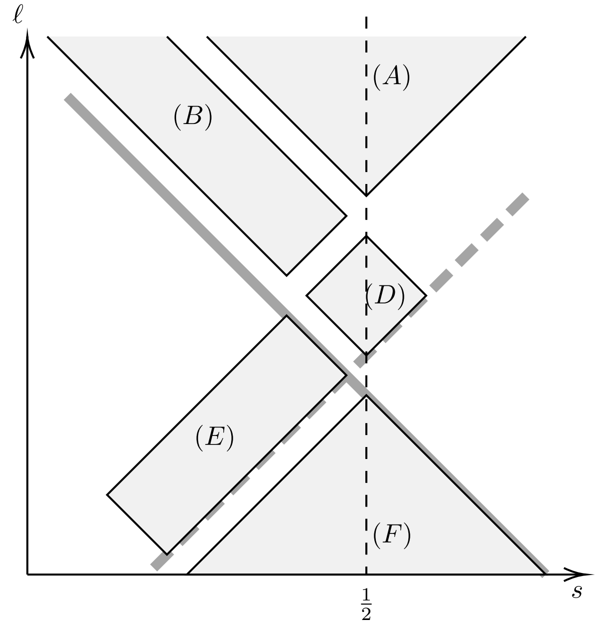
$$(1)\qquad \begin{aligned} k_1 - k_2 + 3 \leqslant \ell \leqslant k_1 + k_2 - 3, \\ \max(k_1, \ell) \leqslant j \leqslant \min(k_2 + \ell + 3, k_1 + k_2 - 3). \end{aligned}$$

The corresponding values of s and ℓ are illustrated in the diagram below. (The "off-centre" regions (B), (E), and the two grey diagonal lines, will be explained shortly.)

2010 *Mathematics Subject Classification.* 11F46; 11F67, 11R23.

Key words and phrases. p-adic L-functions, higher Coleman theory, Siegel Shimura varieties.

Supported by ERC Consolidator grant "Shimura varieties and the BSD conjecture" (D.L.) and Royal Society Newton International Fellowship NIF\R1\202208 (O.R.). This material is based upon work supported by the National Science Foundation under Grant No. DMS-1928930 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2023 semester.



We shall now consider the case when π and σ vary through p-adic families. We consider Coleman families $\underline{\pi}$ for GSp_4 (over some 2-dimensional affinoid space $U \subset \mathcal{W} \times \mathcal{W}$, where $\mathcal{W}$ is the space of characters of $\mathbf{Z}_p^\times$), and similarly $\underline{\sigma}$ for GL_2, over a 1-dimensional affinoid $U' \subset \mathcal{W}$.

Following [LZ20b] and [LR23], we may conjecture that there exist 3 different p-adic L-functions in $\mathcal{O}(U \times U' \times \mathcal{W})$, denoted by $\mathcal{L}^{\spadesuit}(\underline{\pi} \times \underline{\sigma}, -)$ for $\spadesuit \in \{(A), (D), (F)\}$, whose values at integer points (k_1, k_2, ℓ, j) satisfying the inequalities (1) interpolate the corresponding complex L-values. (These depend on various auxiliary data, which we suppress for now).

In [LZ21], building on the earlier work [LPSZ21], we proved a weakened form of this conjecture for region (F): we constructed a p-adic L-function over a codimension-1 subspace of the parameter space $U \times U' \times \mathcal{W}$, interpolating L-values in region (F) and lying at the "right-hand edge" of the critical strip. Thus, for each (k_1, k_2, ℓ) such that $\ell \leqslant k_1 - k_2 + 1$, our p-adic L-function captures just one among the (possibly) many critical values of the L-function of the weight (k_1, k_2, ℓ) specialisation of $\underline{\pi} \times \underline{\sigma}$. This corresponds to the solid grey diagonal line in the above figure. We also showed that certain (non-critical) values of this p-adic L-function, corresponding to the elongation of the diagonal line to meet region (E), were related to syntomic regulators of Euler system classes constructed in [HJS20]; the region (E) in the above diagram is precisely the range of weights in which the geometric Euler system classes of *op.cit.* are defined.

Note 1.1. We would also expect a second Euler system construction for weights in region (B), but this is only conjectural at present. $\diamond$

The goal of this paper is to prove the analogue for region (D) of the first main result proved for region (F) in [LZ21]. That is, we define a p-adic L-function interpolating L-values along the "lower right edge" of region (D), i.e. for (k_1, k_2, ℓ, j) satisfying the conditions

$$k_1 - k_2 + 3 \leqslant \ell \leqslant k_1, \qquad j = k_2 + \ell - 3 \quad \left(\Leftrightarrow\ s = \tfrac{\ell - k_1 + k_2 - 2}{2}\right).$$

So this p-adic L-function again lives over a codimension 1 subspace of the 4-dimensional parameter space, but a different one from that of [LZ21]: it is indicated by the dotted grey line in the figure. We conjecture, but do not prove here, a relation between this new p-adic L-function and syntomic regulators in region (E); we hope to return to this in a subsequent work.

Remark 1.2. Both in the present paper and in [LZ21], the reason why we lose one variable in the construction is that we do not know how to work with *nearly-holomorphic* modular forms in the framework of higher Coleman theory. More precisely, L-values anywhere in region (F), and in the "lower half" of region (D), can be interpreted algebraically via cup-products in coherent cohomology; but the Eisenstein series appearing in these expressions are only holomorphic if s lies at the upper or lower limit of the allowed range – otherwise, they are nearly-holomorphic but not holomorphic. We are optimistic that future developments in higher Hida/Coleman theory may circumvent this barrier, allowing the construction of p-adic L-functions over the full 4-dimensional parameter space with interpolating properties in region (F) or region (D). ◇

The main result. It is convenient to re-index the weights by setting $(r_1, r_2) = (k_1 - 3, k_2 - 3)$; for region (D) to be non-empty we need $r_1 \geqslant r_2 \geqslant 0$, and in this case, (r_1, r_2) is the highest weight of the algebraic representation of GSp_4 for which π is cohomological.

To define the (imprimitive) p-adic L-function, we need to consider the following objects:

- A set of local conditions encoded in terms of the local data γ_S, introduced in §5.1 and §5.4, and appearing in the factor $Z_S(\pi_P \times \sigma_Q, \gamma_S)$.
- A degree eight Euler factor $\mathcal{E}^{(D)}(\pi_P \times \sigma_Q)$, where π_P (resp. σ_Q) stands for the specialization of $\underline{\pi}$ (resp. $\underline{\sigma}$) at the point P (resp. Q).
- The completed (complex) L-function $\Lambda(\pi_P \times \sigma_Q, s)$.
- A basis $\underline{\xi} \otimes \underline{\eta}$ of the space $S^1(\underline{\pi}) \otimes S^1(\underline{\sigma})$, as introduced in [LZ21, Def. 10.4.1]. The p-adic L-function does depend on that choice.
- The complex (resp. p-adic) period $\Omega_\infty(\pi_P, \sigma_Q)$ (resp. $\Omega_P(\pi_P, \sigma_Q)$), depending also on the specialization $\xi_P \otimes \eta_Q$ of the canonical differential $\underline{\xi} \otimes \underline{\eta}$ at (P, Q).
- The Gauss sum attached to χ_σ^{-1}, denoted by $G(\chi_\sigma^{-1})$.

Further, we need to introduce the notion of *nice critical point*. We say a point (P, Q) of $U \times U'$ is nice if $P = (r_1, r_2)$ and $Q = (\ell)$ are integer points, with P nice for $\underline{\pi}$ and Q nice for $\underline{\sigma}$, according to the definitions of Section 5. Further, we say (P, Q) is nice critical if we also have $r_1 - r_2 + 3 \leq \ell \leq r_1 + 3$.

The main theorem we prove in this note, using in a crucial way the algebraicity result of [LR23], is the following.

Theorem 1.3. *There exists a p-adic L-function $\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})$ satisfying the following interpolation property: if (P, Q) is nice critical, then*

$$\frac{\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})(P, Q)}{\Omega_p(\pi_P, \sigma_Q)} = Z_S(\pi_P \times \sigma_Q, \gamma_S) \cdot \mathcal{E}^{(D)}(\pi_P \times \sigma_Q) \cdot \frac{G(\chi_\sigma^{-1}) \Lambda(\pi_P \times \sigma_Q, \frac{\ell - k_1 + k_2 - 2}{2})}{\Omega_\infty(\pi_P, \sigma_Q)},$$

where (k_1, k_2, ℓ) are such that π_P has weight (k_1, k_2) and σ_Q has weight ℓ.

The approach we follow to establish the theorem is the following:

(1) Use results of Harris and Su to express the automorphic period to be computed as a cup product in the coherent cohomology of a Shimura variety associated with $\mathrm{GL}_2 \times \mathrm{GL}_2$. (This has already been carried out in [LR23].)
(2) Use higher Coleman theory to reinterpret the cup product in terms of a pairing in coherent cohomology over certain strata in the adic Shimura varieties.
(3) Use the families of automorphic forms $\underline{\pi}$ and $\underline{\sigma}$ in order to define the p-adic L-function $\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma}; \underline{\xi})$.
(4) Derive an interpolation formula at critical points using the compatibility of the cup-product with specialisation.

Remark 1.4. For this specific value $s = \frac{\ell - k_1 + k_2 - 2}{2}$, we can write $L(\pi_P \times \sigma_Q, s) = L(V, 0)$, where V is the Galois representation $V(\pi_P) \otimes V(\sigma_Q)(k_2 + \ell - 3)$. This Galois representation always has one of its Hodge–Tate weights equal to 0, which gives an intuitive explanation of why it should be "easier" to interpolate L-values along this subspace of the parameter space rather than over the entire 4-dimensional parameter space incorporating arbitrary cyclotomic twists.

If we specialise at a fixed P, giving a one-variable p-adic L-function $L^{\mathrm{imp}}_{p,\gamma_S}(\pi \times \underline{\sigma})$ associated to a fixed π and a GL_2 family $\underline{\sigma}$, and we choose this $\underline{\sigma}$ to be a family of ordinary CM forms (arising from an imaginary quadratic field K in which p is split), then L-values interpolated by $\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})$ can be interpreted as values of the L-function of π twisted by Grössencharacters of K; and the restriction on the value of s implies that the Grössencharacters arising have infinity-types of the form $(n, 0)$. We expect that this L-function should have an interpretation as a "$\mathfrak{p}$-adic L-function" , interpolating twists by characters of the ray class group of K modulo $\mathfrak{p}^\infty$, for a specific choice of prime $\mathfrak{p}$ above p; this will be pursued in more detail elsewhere. ◇

Connection with other works. In [LZ20b], the authors work in the setting of cusp forms for the larger group $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ and conjecture the existence of 6 different p-adic L-functions interpolating Gross–Prasad periods, corresponding to the 'sign +1' regions (a), (a'), (c), (d), (d') and (f) in the diagrams of *op.cit.*. The case of region (f) was covered by the work of Loeffler–Pilloni–Skinner–Zerbes [LPSZ21] (see also [LZ21]) using higher Hida and Coleman theory, and the p-adic L-function for region (c) was announced by Bertolini–Seveso–Venerucci, also using tools from coherent cohomology.

If one formally replaces one of the two cusp forms by an Eisenstein series, then the Gross–Prasad period becomes Novodvorsky's integral computing the degree 8 L-function for $\mathrm{GSp}_4 \times \mathrm{GL}_2$; and regions $(a), (b), (d), (e), (f)$ correspond to the regions (A), (B), (D), (E), (F) of the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ figure above (while the arithmetic meaning of the remaining regions $(a'), (b'), (d'), (c)$ is less clear in this case). The methods we develop in the present work for region (D) can be straightforwardly modified to interpolate $\mathrm{GSp}_4 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ Gross–Prasad periods along one edge of region (d) (and its mirror-image (d')).

For weights in the "off-centre" regions (B) and (E), the complex L-value $L(\pi \times \sigma, s)$ vanishes to order precisely 1, due to the shape of the archimedean Γ-factors. Beilinson's conjecture predicts the existence of canonical motivic cohomology classes whose complex regulators are related to $L'(\pi \times \sigma, s)$; and we expect the images of these classes in p-adic étale cohomology to form Euler systems. For weights in region (E), an Euler system has been obtained in recent work of Hsu, Jin and Sakamoto [HJS20]; and Zerbes and the first author showed in [LZ20a] that the syntomic regulators of these classes are related to values (outside its domain of interpolation) of the p-adic L-function interpolating critical values in region (F). In the last section of this article, we discuss the kind of reciprocity law one can expect relating the cohomology classes of [HJS20] with the p-adic L-function of this article. We hope to come back to this question in a forthcoming work.

Note that this paper requires as an essential input the computations of [LR23], where we compute the local integrals both at the archimedean and the p-adic places. We expect to be able to extend the construction to the whole region (D), adding the *missing* variable once we get a better comprehension of higher Coleman theory for nearly holomorphic modular forms.

Aside from the works listed above, the only other work we know of which treats p-adic interpolation of $\mathrm{GSp}_4 \times \mathrm{GL}_2$ L-values is the PhD thesis of M. Agarwal [Aga07]. Agarwal's construction gives a one-variable p-adic L-function, which appears to correspond to the restriction of our 3-variable function to the line where $k_1 = k_2 = \ell = k$ for a parameter k, although his methods are very different from ours (using an Eisenstein series on the unitary group $U(3,3)$).

Acknowledgements. The authors would like to thank Sarah Zerbes for informative conversations related to this work.

2. Setup: groups and Hecke parameters

2.1. **Groups.** We denote by G the group scheme GSp_4 (over $\mathbf{Z}$), defined with respect to the anti-diagonal matrix $J = \begin{pmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{pmatrix}$; and we let ν be the multiplier map $G \to \mathbf{G}_m$. We define $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, which we embed into G via the embedding

$$\iota : \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right] \mapsto \begin{pmatrix} a &&& b \\ & a' & b' & \\ & c' & d' & \\ c &&& d \end{pmatrix}.$$

We sometimes write h_i for the i-th GL_2 factor of H. We write T for the diagonal torus of G, which is contained in H and is a maximal torus in either H or G.

2.2. **Parabolics.** We write B_G for the upper-triangular Borel subgroup of G, and P_{Si} and P_{Kl} for the standard Siegel and Klingen parabolics containing B, so

$$P_{\mathrm{Si}} = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ && \star & \star \\ && \star & \star \end{pmatrix}, \qquad P_{\mathrm{Kl}} = \begin{pmatrix} \star & \star & \star & \star \\ & \star & \star & \star \\ & \star & \star & \star \\ &&& \star \end{pmatrix}.$$

We write $B_H = \iota^{-1}(B_G) = \iota^{-1}(P_{\mathrm{Si}})$ for the upper-triangular Borel of H.

We have a Levi decomposition $P_{\mathrm{Si}} = M_{\mathrm{Si}} N_{\mathrm{Si}}$, with $M_{\mathrm{Si}} \cong \mathrm{GL}_2 \times \mathrm{GL}_1$, identified as a subgroup of G via

$$(A, u) \mapsto \begin{pmatrix} A & \\ & uA' \end{pmatrix}, \qquad A' \coloneqq \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} {}^t\!A^{-1} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

In this paper P_{Si} and M_{Si} will be much more important than P_{Kl} and M_{Kl} (in contrast to [LPSZ21]) so we shall often denote them simply by P and M. The intersection $B_M \coloneqq M \cap B_G$ is the standard Borel of M; its Levi factor is T.

2.3. **Flag varieties and Bruhat cells.** We write FL_G for the Siegel flag variety $P\backslash G$, with its natural right G-action. There are four orbits for the Borel B_G acting on FL_G, the *Bruhat cells*, represented by a subset of the Weyl group of G, the *Kostant representatives*, which are the smallest-length representatives of the quotient $W_M\backslash W_G$. We denote these by $w_0, \dots, w_3$; see [LZ21] for explicit matrices. Note that the cell $C_{w_i} = P\backslash P w_i B_G \subset \mathrm{FL}_G$ has dimension $\ell(w_i) = i$.

Remark 2.1. For $g \in G$, we can determine which cell C_{w_i} contains the point $Pg \in \mathrm{FL}_G$ via a criterion in terms of the span of the rows of the bottom left 2×2 submatrix of g, as in Remark 5.1.2 of [LZ21]. $\diamond$

Remark 2.2. Note that C_{w_0} and C_{w_3} are stable under P, while $C_{w_1} \sqcup C_{w_2}$ forms a single P-orbit. $\diamond$

Analogously, for the H-flag variety $\mathrm{FL}_H = B_H\backslash H$, we have 4 Kostant representatives $w_{00} = \mathrm{id}$, $w_{10} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{id}\right)$, similarly w_{01}, w_{11} (with the cell $C_{w_{ij}}$ having dimension $i+j$). (This is the whole of the Weyl group of H, since the Levi subgroup of $M_H = T$ is trivial.)

2.4. **Twisted embeddings.**

Definition 2.3. *Let us write* $\tau = \left(\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, 1\right) \in M(\mathbf{Z})$.

(This was denoted γ in [LPSZ21], but γ was also used for a Satake parameter, so we use a different letter here.) Note that τ represents the unique open T-orbit for the M-flag variety $B_M\backslash M$.

2.5. **Coefficient sheaves.** We retain the conventions about algebraic weights and roots of [LZ21]. In particular, we identify characters of T with triples of integers $(r_1, r_2; c)$, with $r_1 + r_2 = c$ modulo 2 corresponding to $\mathrm{diag}(st_1, st_2, st_2^{-1}, st_1^{-1}) \mapsto t_1^{r_1} t_2^{r_2} s^c$. With our present choices of Borel subgroups, a weight $(r_1, r_2; c)$ is dominant for H if $r_1, r_2 \geq 0$, dominant for M_G if $r_1 \geq r_2$, and dominant for G if both of these conditions hold. (We frequently omit the central character c if it is not important in the context.)

For our further use, we briefly recall the conventions of loc. cit. about sheaves. The Weyl group acts on the group of characters $X^*(T)$ via $(w \cdot \lambda)(t) = \lambda(w^{-1} t w)$. As discussed in loc. cit., we can define explicitly $w_G^{\max}$, the longest element of the Weyl group, as well as $\rho = (2, 1; 0)$, which is half the sum of the positive roots for G. There is a functor from representations of P_G to vector bundles on $X_{G,\mathbf{Q}}$; and we let $\mathcal{V}_\kappa$, for $\kappa \in X^\bullet(T)$ that is M_G-dominant, be the image of the irreducible M_G-representation of highest weight κ. Given an integral weight $\nu \in X^\bullet(T)$ such that $\nu + \rho$ is dominant, we define

$$\kappa_i(\nu) = w_i(\nu + \rho) - \rho, \quad 0 \leq i \leq 3,$$

here as usual ρ is half the sum of the positive roots. These are the weights κ such that representations of infinitesimal character $\nu^\vee + \rho$ contribute to $R\Gamma(S_K^{G,\mathrm{tor}}, \mathcal{V}_\kappa)$; if ν is dominant (i.e. $r_1 \geqslant r_2 \geqslant 0$), they are the weights which appear in the *dual BGG complex* computing de Rham cohomology with coefficients in the algebraic G-representation of highest weight ν.

2.6. **Hecke parameters.** With the notations of the introduction, let π be a cuspidal automorphic representation of G, and let p be a prime. If π_{f} is unramified at p, we write α, β, γ, δ for the Hecke parameters of π'_p, and $P_p(X)$ for the polynomial $(1 - \alpha X) \dots (1 - \delta X)$. The Hecke parameters are algebraic integers over a number field E, and are well-defined up to the action of the Weyl group. They all have complex absolute value $p^{w/2}$, where $w \coloneqq r_1 + r_2 + 3$, and they satisfy $\alpha\delta = \beta\gamma = p^w \chi_\pi(p)$, where $\chi_\pi(p)$ is a root of unity.

Let $\mathrm{Iw}_G(p)$ denote the Iwahori subgroup. We shall consider the following operators in the Hecke algebra of level $\mathrm{Iw}_G(p)$, acting on the cohomology of any of the sheaves introduced before:

- The Siegel operator $\mathcal{U}_{\mathrm{Si}} = [\mathrm{diag}(p, p, 1, 1)]$, as well as its dual $\mathcal{U}'_{\mathrm{Si}} = [\mathrm{diag}(1, 1, p, p)]$.

- The Klingen operator $\mathcal{U}_{\mathrm{Kl}} = p^{-r_2} \cdot [\mathrm{diag}(p^2, p, p, 1)]$, as well as its dual $\mathcal{U}'_{\mathrm{Kl}} = p^{-r_2} \cdot [\mathrm{diag}(1, p, p, p^2)]$.
- The Borel operator $\mathcal{U}_B = \mathcal{U}_{\mathrm{Si}} \cdot \mathcal{U}_{\mathrm{Kl}}$, as well as its dual $\mathcal{U}'_B = \mathcal{U}'_{\mathrm{Si}} \cdot \mathcal{U}'_{\mathrm{Kl}}$.

For a place v of E above p, we shall say that π is Siegel ordinary at p with respect to v if the operator $\mathcal{U}_{\mathrm{Si}}$ on $(\pi'_p)^{\mathrm{Si}(p)}$ has an eigenvalue λ which is a p-adic unit. We define similarly the condition of being Klingen ordinary and we say that π is Borel-ordinary at p if it is both Siegel and Klingen ordinary. The condition of being Siegel ordinary at p may be rephrased by requiring that $v_p(\alpha) = 0$. The condition of being Klingen ordinary is equivalent to $v_p(\alpha\beta) = r_2 + 1$. In this work, however, we will usually consider more relaxed conditions corresponding to being *finite slope*. We explore this issue in the next section.

For a cuspidal automorphic representation σ of GL_2, write $\mathfrak{a}$, $\mathfrak{b}$ for the Hecke parameters of σ'_p. We adopt the convention that $v_p(\mathfrak{a}) \leq v_p(\mathfrak{b})$ and say that σ is Borel-ordinary at p (with respect to v) if $v_p(\mathfrak{a}) = 0$.

2.7. **Slope conditions.** We consider the Hecke operators with the previous normalizations acting on the cohomology of the different sheaves $\mathcal{V}_\kappa$. Thus each operator is "minimally integrally normalised" acting on the classical cohomology (slopes are ≥ 0). Write K^p for some fixed choice of open compact away from p. Then [BP20, Conj. 5.29] predicts lower bounds for the slopes of the Hecke operators acting on the overconvergent cohomology complexes $R\Gamma_w(K^p, \kappa)^{\pm}$ and $R\Gamma(K^p, \kappa, \mathrm{cusp})^{\pm}$, whose precise definitions are given in loc. cit.; and there are similar conjectures for the locally-analytic cohomology complexes.

We compute for various elements $w \in W_G$ the character $w^{-1} w_G^{\max}(\kappa + \rho) - \rho$, and how it pairs with the anti-dominant cocharacters $\mathrm{diag}(1, 1, x, x,)$ and $\mathrm{diag}(1, x, x, x^2)$ defining the operators $\mathcal{U}'_{\mathrm{Si}}$ and $\mathcal{U}'_{\mathrm{Kl}}$. We take $\kappa = \kappa_2 = (r_2 - 1, -r_1 - 3; r_1 + r_2)$, and subtract r_2 from all entries in the bottom row since this is our normalising constant for $\mathcal{U}'_{\mathrm{Kl}}$. We summarize the slope bounds in the following table.

$w =$	id	w_1	(w_2)	w_3
$\mathcal{U}'_{\mathrm{Si}}$	$r_1 + 2$	0	(0)	$r_2 + 1$
$\mathcal{U}'_{\mathrm{Kl}}$	$r_1 - r_2 + 1$	$r_1 - r_2 + 1$	(0)	0

We do not know this conjecture in full, but we do know a weaker statement in which we replace $w^{-1} w_G^{\max}(\kappa_2 + \rho) - \rho$ with $w^{-1} w_G^{\max} \kappa_2$. This gives the following bounds:

$w =$	id	w_1	(w_2)	w_3
$\mathcal{U}'_{\mathrm{Si}}$	$r_1 + 2$	-1	(-1)	$r_2 - 2$
$\mathcal{U}'_{\mathrm{Kl}}$	$r_1 - r_2 + 1$	$r_1 - r_2 + 1$	(-3)	-3

Proposition 2.4. *For the weight $\kappa_2 = (r_2 - 1, -r_1 - 3; r_1 + r_2)$, with $r_1 \geq r_2 \geq 0$, we have the following.*

- *The "small slope" condition $(-, \mathrm{ss}^M(\kappa_2))$ is*
$$\lambda(\mathcal{U}'_{\mathrm{Si}}) < r_1 + 2, \quad \lambda(\mathcal{U}'_{\mathrm{Kl}}) < r_1 - r_2 + 1.$$
- *The "strictly small slope" condition $(-, \mathrm{sss}^M(\kappa_2))$ is*
$$\lambda(\mathcal{U}'_{\mathrm{Si}}) < r_1 + 2, \quad \lambda(\mathcal{U}'_{\mathrm{Kl}}) < r_1 - r_2 - 2.$$

Proof. This follows from the previous tables. □

3. Flag varieties and orbits

Let P denote the Siegel parabolic subgroup of $G = \mathrm{GSp}_4$, and $\mathrm{FL}_G = P \backslash G$ with its natural right G-action.

3.1. **Kostant representatives.** There are four orbits for the Borel B_G acting on FL_G (Bruhat cells), represented by a subset of the Weyl group of G, the *Kostant representatives* (a distinguished set of representatives for the quotient $W_{M_G} \backslash W_G$ where M_G is the Levi of P). We denote these by $w_0, \ldots, w_3$; see [LZ21] for explicit matrices. Note that the cell $C_{w_i} = P \backslash P w_i B_G$ has dimension i.

Remark 3.1. For $g \in G$, we can determine which cell C_{w_i} contains the point $Pg \in \mathrm{FL}_G$ via a criterion in terms of the span of the rows of the bottom left 2×2 submatrix of g, as in Remark 5.1.2 of [LZ21]. ◇

Analogously for H we have 4 Kostant representatives $w_{00} = \mathrm{id}$, $w_{10} = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathrm{id} \right)$, similarly w_{01}, w_{11} (with the cell $C_{w_{ij}}$ having dimension $i + j$). (This is the whole of the Weyl group of H, since the maximal compact of H is abelian and hence the Weyl group of its Levi is trivial).

Remark 3.2. Either for G or for H, each cell will determine a subspace of the Iwahori-level Shimura variety (as an adic space), via pullback along the Hodge–Tate period map. This is the locus where the relative position of the Hodge filtration and level structure on the p-divisible group lies in the given Bruhat cell. In particular, the "smallest" cell (w_0 or w_{00}) corresponds to the multiplicative locus, and the "largest" one to the étale locus. $\diamond$

3.2. **A twisted embedding of flag varieties.** Consider the elements

$$\tau = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{pmatrix} \in M_{\mathrm{Si}}(\mathbf{Z}_p), \qquad \tau^\sharp = \iota(w_{01})^{-1}\tau w_2 \in G(\mathbf{Z}_p).$$

We will consider the translated embedding $\iota^\sharp : H \to G$ given by $h \mapsto \iota(h)\tau^\sharp$. The map $\mathrm{FL}_H \to \mathrm{FL}_G$ induced by $\iota^\sharp$ by construction sends $[w_{01}]$ to $[w_2]$. We also have projection maps $\pi_i : \mathrm{FL}_H \to \mathrm{FL}_{\mathrm{GL}_2} \cong \mathbf{P}^1$, and the product $(\iota^\sharp, \pi_1, \pi_2)$ evidently sends w_{01} to $([w_2], [\mathrm{id}], [w])$ (where the unlabelled w is the GL_2 long Weyl element).

If we equate $(x : y) \in \mathbf{P}^1$ with the orbit $Bg \in B\backslash G$, where g is any invertible matrix of the form $\left(\begin{smallmatrix} \star & \star \\ x & y \end{smallmatrix}\right)$, then $\iota^\sharp$ sends $((x:y),(X:Y))$ to

$$P_{\mathrm{Si}} \cdot \begin{pmatrix} \star & \dots & \dots & \star \\ \star & \dots & \dots & \star \\ -X & Y & \star & \star \\ -y & x & \star & \star \end{pmatrix}.$$

Using this and the explicit description of the Bruhat cells in terms of the bottom left corner of the matrix, we see that:

- the preimage of C^G_{id} is empty;
- the preimage of $C^G_{w_1}$ is the point $((1:0),(0:1))$ (the image of $[w_{10}] \in \mathrm{FL}_H$);
- the preimage of $C^G_{w_2}$ is a copy of the affine line, corresponding to points of the form

$$B_H \left(\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \\ -x & 1 \end{smallmatrix}\right) \right) w_{01}.$$

Proof. The condition for the above matrix to lie in $X^G_{w_2}$ is that $\left(\begin{smallmatrix} -X & Y \\ -y & x \end{smallmatrix}\right)$ be singular, i.e. $Xx = Yy$; and the condition for it to lie in $C^G_{w_2}$ is that the span of the rows not be $(0:1)$, so $X \neq 0$ and $y \neq 0$. So we can wlog take $X = 1$ and $y = 1$, leaving the equation $Y = x$; i.e. our point was $B_H \left(\left(\begin{smallmatrix} \star & \star \\ x & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \star & \star \\ 1 & x \end{smallmatrix}\right) \right) = B_H \left(\left(\begin{smallmatrix} \star & \star \\ x & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \star & \star \\ -x & 1 \end{smallmatrix}\right) \right) w_{01}$. $\square$

Notation. *Recall* $X^G_w = \bigcup_{w' \leqslant w} C^G_{w'}$ *(closed subvariety), and* $Y^G_w = \bigcup_{w' \geqslant w} C^G_{w'}$ *(open subvariety).*

Proposition 3.3. *We have*

$$(\iota^\sharp)^{-1}\left(X^G_{w_2}\right) \cap \pi_2^{-1}\left(Y^{\mathrm{GL}_2}_w\right) = (\iota^\sharp)^{-1}\left(C^G_{w_2}\right) \cap \pi_1^{-1}(C^{\mathrm{id}}_{\mathrm{GL}_2}) \cap \pi_2^{-1}\left(C^{\mathrm{GL}_2}_w\right).$$

(Note $C^{\mathrm{id}}_{\mathrm{GL}_2}$ is not a typo: it denotes $B\backslash B\overline{B}$, the big cell at the origin, cf. [BP20, §3.1].)

Proof. Since the single point $(\iota^\sharp)^{-1}(C^G_{\mathrm{id}} \cup C^G_{w_1}) = [w_{10}]$ does not map to $Y^{\mathrm{GL}_2}_w$ under π_2, we conclude that $(\iota^\sharp)^{-1}\left(X^G_{w_2}\right) \cap \pi_2^{-1}\left(Y^{\mathrm{GL}_2}_w\right)$ is contained in $(\iota^\sharp)^{-1}(C^G_{w_2})$. $\square$

3.3. **Some tubes.** We now work with the flag variety as an adic space and consider various tubes inside $\mathrm{FL}_G \times \mathbf{P}^1 \times \mathbf{P}^1$. As usual $\mathcal{X}^G_w$ denotes the tube of $X^G_{w,\mathbf{F}_p}$ etc.

We shall set

$$\mathtt{Z}_0 = \overline{\mathcal{X}^G_{w_2}} \times \text{everything} \times \overline{\mathcal{Y}^{\mathrm{GL}_2}_w},$$

and

$$\mathtt{U}_0 = \mathcal{Y}^G_{w_2} \times \mathcal{X}^{\mathrm{GL}_2}_{\mathrm{id}} \times \text{everything}.$$

Then $\mathtt{Z}_0$ is closed, $\mathtt{U}_0$ is open, and both are stable under the action of $\mathrm{Iw}_G \times \mathrm{Iw}_{\mathrm{GL}_2} \times \mathrm{Iw}_{\mathrm{GL}_2}$; and $\mathtt{U}_0 \cap \mathtt{Z}_0$ is a partial closure of the (w_2, id, w) Bruhat cell for $G \times \mathrm{GL}_2 \times \mathrm{GL}_2$.

We need to allow smaller "overconvergence radii", for which we use the action of the element $\eta_G = \mathrm{diag}(p^3, p^2, p, 1)$ and its cousin $\eta = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$.

Definition 3.4. *Let us set*

$$\mathtt{Z}_m = \left(\overline{\mathcal{X}^G_{w_2}} \cdot \eta_G^m\right) \times \textit{everything} \times \left(\overline{\mathcal{Y}^{\mathrm{GL}_2}_w} \cdot \eta^{-m} \left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)\right).$$

We have $\mathtt{Z}_0 \supseteq \mathtt{Z}_1 \supseteq \mathtt{Z}_2 \dots$, by [BP20, Lemma 3.23], and $\mathtt{Z}_m$ is stable under $\mathrm{Iw}(p^t)$ for $t \geqslant 3m+1$.

Recall that we have explicit coordinates on the Zariski-open neighbourhood $U_{w_2} = P\backslash P\overline{P}w_2$ of $[w_2] \in \mathrm{FL}_G$, given by $P\backslash P \begin{pmatrix} 1 & & & \\ & 1 & & \\ x & y & 1 & \\ z & x & & 1 \end{pmatrix} w_2$, as usual. Then one computes that

$$\overline{\mathcal{X}^G_{w_2}} \cap U^{\mathrm{an}}_{w_2} = \left\{ (x,y,z) : x \notin \mathcal{B}_0 \text{ or } y \notin \mathcal{B}_0 \text{ or } z \in \overline{\mathcal{B}}_0^\circ \right\},$$

and η_G preserves $U^{\mathrm{an}}_{w_2}$ and acts in these coordinates via $(x,y,z) \mapsto (p^{-1}x, p^{-3}y, pz)$. Thus

$$\mathtt{Z}^G_m \cap U^{\mathrm{an}}_{w_2} = \left\{ (x,y,z) : x \notin \mathcal{B}_{-m} \text{ or } y \notin \mathcal{B}_{-3m} \text{ or } z \in \overline{\mathcal{B}}_m^\circ \right\},$$

and a similar computation identifies $\overline{\mathcal{Y}^{\mathrm{GL}_2}_w}$ with $\overline{\mathcal{B}}_0$, and $\overline{\mathcal{Y}^{\mathrm{GL}_2}_w} \cdot \eta^{-m} \left(\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ with $\overline{\mathcal{B}}_m + \mathbf{Z}_p$.

On the other hand, we can define

$$\mathtt{U}_n = \left(\mathcal{Y}^G_{w_2} \cdot \eta_G^{-m} N_{B_G}(\mathbf{Z}_p)\right) \times \left(\mathcal{X}^{\mathrm{GL}_2}_{\mathrm{id}} \cdot \eta^n\right) \times \text{everything}.$$

Again, this is stable under $\mathrm{Iw}(p^t)$ for $t \geqslant n+1$, and the G-part of the locus is given by

$$\mathtt{U}^G_n \cap U^{\mathrm{an}}_{w_2} = \left\{ (x,y,z) : x \in \mathcal{B}_n + \mathbf{Z}_p, y \in \mathcal{B}_{3n} + \mathbf{Z}_p \right\},$$

with no condition on z; and the projection to the first GL_2 coordinate is just $\mathcal{B}_n^\circ$.

Lemma 3.5. *The intersection $\mathtt{Z}^G_m \cap \mathtt{U}^G_n$ is contained in $U^{\mathrm{an}}_{w_2}$, for all $m, n \geqslant 0$.*

Proof. It suffices to check this for $(m,n) = (0,0)$; see Lemma 3.21 of [BP20]. □

3.4. **Pullback to H.** Guided by the zeta-integral computations of [LR23], we shall consider the map

$$\iota^{\sharp\sharp} : \mathrm{FL}_H \to \mathrm{FL}_G \times \mathrm{FL}_H, \qquad h \mapsto \left(\iota^\sharp(h), h_1 \begin{pmatrix} p^t & \\ & 1 \end{pmatrix}, h_2\right).$$

for some $t \geqslant 1$.

Proposition 3.6. *If $m > 3n \geqslant 0$, then*

$$(\iota^{\sharp\sharp})^{-1}(\mathtt{Z}_m \cap \mathtt{U}_n) = (\iota^{\sharp\sharp})^{-1}(\mathtt{Z}_m),$$

and in particular this preimage is closed in FL_H.

Proof. We know that the pullback of $\mathtt{Z}_0$ is contained in the big cell, so we can compute it in coordinates. We find that the inequalities on (z_1, z_2) for it to land in $\mathtt{Z}_m$ are:

$$z_1 + z_2 \in \overline{\mathcal{B}^\circ_m}, \qquad z_2 \in \overline{\mathcal{B}_m} + \mathbf{Z}_p.$$

For $\mathtt{Z}_m \cap \mathtt{U}_n$ we add the extra inequalities

$$z_2 \in \mathcal{B}_{3n} + \mathbf{Z}_p, \qquad p^t z_1 \in \mathcal{B}^\circ_n.$$

If $m > 3n$ then the latter equations are a consequence of the former. □

3.5. **Periods maps and overconvergent cohomology.** We consider the analytifications $\mathcal{S}^{\mathrm{an}}_{G,K} = (S_K \times \mathrm{Spec}(\mathbf{Q}_p))^{\mathrm{an}}$ and $\mathcal{S}^{\mathrm{tor}}_{G,K} = (S^{\mathrm{tor}}_{G,K} \times \mathrm{Spec}(\mathbf{Q}_p)^{\mathrm{an}}$, and similarly for H and $G \times H$ (denoted always by calligraphic letters). Write $\mathcal{S}^{\mathrm{tor}}_{G,K^p}$ for the perfectoid space $\lim_{\leftarrow K_p} \mathcal{S}^{\mathrm{tor}}_{G,K^pK_p}$, which allows us to consider the Hodge–Tate period map

$$\pi^{\mathrm{tor}}_{\mathrm{HT},G} : \mathcal{S}^{\mathrm{tor}}_{G,K^p} \longrightarrow \mathtt{FL}_G,$$

which for every open compact $K_p \subset G(\mathbf{Q}_p)$ descends to a map of topological spaces (c.f. [BP20, §4.5]

$$\pi^{\mathrm{tor}}_{\mathrm{HT},G,K_p} : \mathcal{S}^{\mathrm{tor}}_{G,K^pK_p} \longrightarrow \mathtt{FL}_G / K_p.$$

There are also analogous maps for H and $G \times H$.

Then [LZ21, Thm. 6.2.1] holds in the same way, replacing the level structure $K^H_\diamond$ by $K^H_\triangle$, defined as follows

Definition 3.7. *Let $K^H_\triangle(p^t) = K^H_{\mathrm{Iw}}(p^t) \cap \tau^\sharp K^G_{\mathrm{Iw}}(p^t)(\tau^\sharp)^{-1}$. It is concretely given by*

$$K^H_\triangle(p^t) = \{h \in H(\mathbf{Z}_p) \mid h = \left(\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, \begin{pmatrix} z & 0 \\ 0 & x \end{pmatrix}\right) \mod p^t \textit{ for some } x\}.$$

The choices of neighbourhoods we have made are sufficient to get the maps working, including the compatibility with the classical cohomology via compact-support cohomology of $\mathtt{Z}_m$. Write $\mathtt{Z}_m^H$ for the corresponding preimage in H. Let $\mathcal{U}_n^G \subset \mathcal{S}_{G\times H,\mathrm{Iw}}(p^t)$ and $\mathcal{Z}_m^H \subset \mathcal{U}_n^H \subset \mathcal{S}_{H,\triangle}(p^t)$ denote the preimages of the subsets $\mathtt{U}_n^G \subset \mathrm{FL}^G$ and $\mathtt{Z}_m^H \subset \mathtt{U}_n^H \subset \mathrm{FL}^H$ under $\pi_{\mathrm{Iw}}^{G\times H}$ and $\pi_{\triangle}^H$.

$$\begin{array}{ccc}
R\Gamma(\mathcal{U}_0^G,\mathcal{V}) & \longrightarrow & R\Gamma(\mathcal{S}_{H,\triangle}^{\mathrm{tor}}(p^t),(\iota^\sharp)^*(\mathcal{V})) \\
\uparrow & & \uparrow \\
R\Gamma_{\mathcal{Z}_m}(\mathcal{U}_0^G,\mathcal{V}) & \longrightarrow & R\Gamma_{\mathcal{Z}_m^H}(\mathcal{S}_{H,\triangle}^{\mathrm{tor}}(p^t),(\iota^\sharp)^*(\mathcal{V})) \\
\downarrow & \nearrow & \\
R\Gamma_{\mathcal{Z}_m}(\mathcal{U}_n^G,\mathcal{V}) & &
\end{array}$$

Here, the horizontal maps correspond to $(\iota^\sharp)^*$, while the vertical ones are the usual restriction and corestriction.

4. Branching laws and sheaves of distributions

In this section, we introduce the necessary tools to p-adically interpolate the pairing of the previous section. We keep the notations of [LZ21, §8] and review some of the more relevant results of loc. cit., focusing on the changes we need in our setting. Note in particular that the discussions and results of §6 hold verbatim, with the obvious changes in Prop. 6.4.1.

Along this section, we will frequently consider the projections of the embedding $\iota^{\sharp\sharp}$ on each factor: the first component corresponds to $\iota^\sharp : \mathrm{FL}_H \to \mathrm{FL}_G$, and the second, referred as ι_p, is the map

$$\iota_p : \mathrm{FL}_H \to \mathrm{FL}_H, \qquad (h_1,h_2) \mapsto \left(h_1 \begin{pmatrix} p^t & \\ & 1 \end{pmatrix}, h_2\right).$$

We also write $\upsilon = \upsilon(p^t) = \left(\begin{pmatrix} p^t & \\ & 1 \end{pmatrix}, 1\right) \in M(\mathbf{Z})$.

4.1. Torsors.

The map $x \mapsto x^{-1} : G \to \mathrm{FL}_G$ allows us to regard G as a right P_G-torsor over FL_G, and similarly to regard $G/N_G \to \mathrm{FL}_G$ as a right M_G-torsor. We consider their analytifications

$$\mathtt{P}^G : \mathcal{G} \to \mathtt{FL}_G \quad \text{and} \quad \mathtt{M}^G : \mathcal{G}/\mathcal{N}_G \to \mathtt{FL}_G$$

which are torsors over $\mathtt{FL}_G$ under the analytic groups $\mathcal{P}_G$ and $\mathcal{M}_G$ respectively. We similarly define torsors over the flag varieties H, H_1 and H_2.

Definition 4.1. *Define $\mathcal{P}_{\mathrm{HT}}^G$ and $\mathcal{M}_{\mathrm{HT}}^G$ to be the pullbacks via π_{HT}^G of the torsors $\mathtt{P}^G$ and $\mathtt{M}^G$; there are right torsors over $\mathcal{S}_{G,\mathrm{Iw}}(p^t)$ for the groups $\mathcal{P}_G$ and $\mathcal{M}_G$. We similarly define $\mathcal{P}_{\mathrm{HT}}^H$ and $\mathcal{M}_{\mathrm{HT}}^H$, $\mathcal{P}_{\mathrm{HT}}^{H_i}$ and $\mathcal{M}_{\mathrm{HT}}^{H_i}$ for $i=1,2$.*

Definition 4.2. *For $n>0$, let $\mathcal{M}_{G,n}^1$ be the group of elements which reduce to the identity modulo p^n. Define*

$$\mathcal{M}_{G,n}^{\square} = \mathcal{M}_{G,n}^1 \cdot B_{M_G}(\mathbf{Z}_p),$$

which is an affinoid analytic subgroup containing $\mathrm{Iw}_{M_G}(p^n)$. A similar definition applies to $M_H = T$; we write the group as $\mathcal{T}_n^{\square} = T(\mathbf{Z}_p)\mathcal{T}_n^1$.

Consider in the same way

$$\mathcal{T}_n^{\diamond} = \{\mathrm{diag}(t_1,t_2,\nu t_2^{-1},\nu t_1^{-1}) \in \mathcal{T}_n^{\square} \,:\, t_1 - t_2 \in \mathcal{B}_n\}.$$

As in [LZ21, Prop. 7.2.5], we also consider the étale torsors $\mathcal{M}_{\mathrm{HT},n}^G$, $\mathcal{M}_{\mathrm{HT},n,\mathrm{Iw}}^G$ and $\mathcal{M}_{\mathrm{HT},n,\diamond}^H$ arising as the reduction of structure of the torsors $\mathcal{M}_{\mathrm{HT}}^G$ over $\mathcal{U}_n^G$, $\mathcal{M}_{\mathrm{HT}}^H$ over $\mathcal{U}_{\mathrm{Iw},n}^H$ and $\mathcal{M}_{\mathrm{HT}}^H$ over $\mathcal{U}_n^H$, respectively.

Proposition 4.3. *We have an equality of $\mathcal{M}_{G,n}^{\square}$-torsors over $\mathcal{U}_{n,\diamond}^H$:*

$$(\iota^\sharp)^*(\mathcal{M}_{\mathrm{HT},n,\mathrm{Iw}}^G) = \mathcal{M}_{\mathrm{HT},n,\diamond}^H \times^{[\mathcal{T}_n^{\diamond},\tau]} \mathcal{M}_{G,n}^{\square},$$

where we regard $\mathcal{T}_n^{\diamond}$ as a subgroup of $\mathrm{Iw}_{M_G}(p^t)\mathcal{M}_{G,n}^1$ via conjugation by τ.

Proof. This follows by checking the analogous statement on the flag variety, noting that there is a commutative diagram of adic space:

$$\begin{array}{ccc} K^H_{\triangle}(p^t)\mathcal{H}^1_n & \longrightarrow & K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_n \\ \downarrow & & \downarrow \\ \mathcal{B}^H\backslash\mathcal{B}^H w_{01} K^H_{\triangle}(p^t)\mathcal{H}^1_n & \longrightarrow & \mathcal{P}^G\backslash\mathcal{P}^G w_2 K^G_{\mathrm{Iw}}(p^t)\mathcal{G}^1_n. \end{array}$$

Here, the vertical maps are given by $h \mapsto \mathcal{B}^H\backslash\mathcal{B}^H h^{-1}$ on the left, and $g \mapsto \mathcal{P}^G\backslash\mathcal{P}^G w_2 g^{-1}$ on the right; the lower horizontal map is $\iota^\sharp$ is $\mathcal{B}^H h \mapsto \mathcal{P}^G h\tau w_2$, and the map along the top making the diagram commute is $h \mapsto (\tau^\sharp)^{-1} h \tau^\sharp$.

Then we may conclude as in [LZ21, Prop. 7.2.7]. □

A straightforward adaptation of these techniques can be applied to the second factor ι_p, yielding to an equality of $\mathcal{M}^\square_{H,n}$-torsors over $\mathcal{U}^H_{n,\diamond}$,

$$(\iota_p)^*(\mathcal{M}^H_{\mathrm{HT},n,\mathrm{Iw}}) = \mathcal{M}^H_{\mathrm{HT},n,\diamond} \times^{[\mathcal{T}^\diamond_n,\upsilon]} \mathcal{M}^\square_{H,n},$$

where we regard $\mathcal{T}^\diamond_n$ as a subgroup of $\mathrm{Iw}_{M_H}(p^t)\mathcal{M}^1_{H,n}$ via conjugation by υ. Observe that conjugation by υ does not introduce denominators in any element of M_H, and hence the previous objects are well defined.

4.2. **Analytic characters and analytic inductions.**

Definition 4.4. *Let $n \in \mathbf{Q}_{>0}$. We say that a continuous character $\kappa : \mathbf{Z}_p^\times \to A^\times$, for (A, A^+) a complete Tate algebra, is n-analytic if it extends to an analytic A-valued function on the affinoid adic space*

$$\mathbf{Z}_p^\times \cdot \mathcal{B}_n \subset \mathbf{G}_m^{\mathrm{ad}}.$$

This definition extends to characters $T(\mathbf{Z}_p) \to A^\times$: the n-analytic characters are exactly those which extend to $\mathcal{T}^\square_n$.

Let $n_0 > 0$ and assume that $\kappa_A : T(\mathbf{Z}_p) \to A^\times$ is an n_0-analytic character. For $? \in \{G, H\}$ and $n \geq n_0$, let $\mathcal{M}^1_{?,n}$ be the affinoid subgroup of $\mathcal{M}_?$ defined above, and let B_{M_G} be the Borel of $M_?$.

Definition 4.5. *For $n \geq n_0$, define*

$$\begin{aligned} V^{n-\mathrm{an}}_{G,\kappa_A} &= \mathrm{anInd}^{(\mathcal{M}^\square_{G,n})}_{(\mathcal{M}^\square_{G,n}\cap\mathcal{B}_G)}(w_{0,M_?}\kappa_A) \\ &= \Big\{ f \in \mathcal{O}(\mathcal{M}^\square_{G,n})\hat{\otimes}A \,:\, f(mb) = (w_{0,M}\kappa_A)(b^{-1})f(m),\ \forall m \in \mathcal{M}^\square_{G,n},\ \forall b \in \mathcal{M}^\square_{G,n} \cap \mathcal{B}_G \Big\}. \end{aligned}$$

We define a left action of $\mathcal{M}^\square_{G,n}$ on $V^{n-\mathrm{an}}_{G,\kappa_A}$ by $(h \cdot f)(m) = f(h^{-1}m)$.

Write $D^{n-\mathrm{an}}_{G,\kappa_A}$ for the dual space, and $\langle\cdot,\cdot\rangle$ for the pairing between these; we equip $D^{n-\mathrm{an}}_{G,\kappa_A}$ with a left action of the same group $\mathcal{M}^\square_{G,n}$, in such a way that $\langle h\mu, hf\rangle = \langle\mu, f\rangle$.

As shown in [LZ21, Prop. 8.2.2], for a character κ of the form $(\rho_1, \rho_2; \omega)$ the action of $(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \nu)$ on $f \in \mathcal{O}(\mathcal{B}_n)\hat{\otimes}A$ is given by

$$((\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \nu)f)(z) = f\Big(\frac{az - c}{-bz + d}\Big)(-bz + d)^{\rho_1-\rho_2}(ad - bc)^{\rho_2}\nu^{(\omega-\rho_1-\rho_2)/2}.$$

4.3. **Branching laws in families.**

Recall that for a Tate algebra A endowed with an n_0-analytic character $\kappa_A : T(\mathbf{Z}_p) \to A^\times$ as above, and additionally with a character $\lambda : (1 + \mathcal{B}_n)^\times \to A^\times$, we may define the krakenfish function as $\mathcal{K}^\lambda(z) = \lambda(1 + z)$, viewed as an element of $V^{n-\mathrm{an}}_{G,\kappa_A}$.

The following lemma is analogous to [LZ21, Lemma 8.3.2], but recall that now the objects involved in the definition of $\mathcal{T}^\diamond_n$ are different.

Lemma 4.6. *The function $\mathcal{K}^\lambda$ is an eigenvector for $(\tau^\sharp)^{-1}\mathcal{T}^\diamond_n\tau^\sharp$, with eigencharacter $w_{0,M}\kappa_A + (\lambda, -\lambda; 0)$.*

Proof. This follows from the computations of [LZ21, §8.3] by noting that τ lies in the Siegel parabolic and that only the projection to the Levi subgroup matters for this computation. □

The following result is a straightforward consequence of the previous lemma.

Proposition 4.7. *Pairing with $\mathcal{K}^\lambda$ defines a homomorphism of $\mathcal{T}_n^\diamond$-representations*

$$(\iota^\sharp)^*(D^{n-\mathrm{an}}_{G,\kappa_A}) \longrightarrow D^{n-\mathrm{an}}_{H,w_{01,M}\kappa_A+(\lambda,-\lambda;0)}.$$

4.4. **Labelling of weights.** As above, let (A, A^+) be a Tate algebra over $(\mathbf{Q}_p, \mathbf{Z}_p)$. Given a weight $\nu_A : T(\mathbf{Z}_p) \to A^\times$ for some coefficient ring A, we may define $\kappa_A : T(\mathbf{Z}_p) \to A^\times$ by

$$\kappa_A = -w_{0,M} w_2(\nu + \rho) - \rho.$$

If ν_A is $(\nu_1, \nu_2; \omega)$ for some $\nu_i, \omega : \mathbf{Z}_p^\times \to A^\times$, then $\kappa_A = (\nu_1, -2-\nu_2; \omega)$. Its Serre dual is $\kappa'_A = (\kappa_A + 2\rho_{\mathrm{nc}})^\vee$. This can be written as $(\nu_2 - 1, -3 - \nu_1; c) = w_2(\nu_A + \rho) - \rho$.

4.5. **Sheaves on G.** Let $1 \le n < t$ be integers. The following definition is just [LZ21, Def. 9.2.1].

Definition 4.8. *The sheaf $\mathcal{V}^{n-\mathrm{an}}_{G,\nu_A}$ over $\mathcal{U}^G_n$ is given by the product*

$$\mathcal{V}^{n-\mathrm{an}}_{G,\nu_A} = \mathcal{M}^G_{\mathrm{HT},n,\mathrm{Iw}} \times^{M^\square_{G,n}} V^{n-\mathrm{an}}_{G,\kappa_A}.$$

We define similarly another sheaf $\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A}$ as

$$\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A} = \mathcal{M}^G_{\mathrm{HT},n,\mathrm{Iw}} \times^{M^\square_{G,n}} D^{n-\mathrm{an}}_{G,(\kappa_A+2\rho_{\mathrm{nc}})}.$$

As discussed in loc. cit., the sheaves $\mathcal{V}^{n-\mathrm{an}}_{G,\nu_A}$ and $\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A}$ are sheaves of A-modules compatible with base-change in A. If $A = \mathbf{Q}_p$ and $\nu_A = (r_1, r_2; c)$ for integers $r_1 \ge r_2 \ge -1$, we have classical comparison maps

$$\mathcal{V}_{G,\kappa_1} \hookrightarrow \mathcal{V}^{n-\mathrm{an}}_{G,\nu_A}, \quad \mathcal{D}^{n-\mathrm{an}}_{G,\nu_A} \twoheadrightarrow \mathcal{V}_{G,(\kappa_A+2\rho_{\mathrm{nc}})^\vee} = \mathcal{V}_{G,\kappa_2}.$$

4.6. **Sheaves on H.** We mimic the same definitions for H, using now the element $w_{01} \in^M W_H$ in place of w_2. Given an n-analytic character τ_A, we define $\kappa^H_A = -\tau_A - 2\rho_H$, and we set

$$\mathcal{V}^{n-\mathrm{an}}_{H,\diamond,\nu_A} = \mathcal{M}^H_{\mathrm{HT},n,\diamond} \times^{\mathcal{T}_n^\diamond} V^{n-\mathrm{an}}_{H,\kappa^H_A}$$

and

$$\mathcal{D}^{n-\mathrm{an}}_{H,\diamond,\tau_A} = \mathcal{M}^H_{\mathrm{HT},n,\mathrm{Iw}} \times^{\mathcal{T}_n^\diamond} D^{n-\mathrm{an}}_{H,(\kappa^H_A+2\rho_H)}.$$

4.7. **Branching for sheaves.**

Definition 4.9. *We say that A-valued, n-analytic characters ν_A and τ_A of $T(\mathbf{Z}_p)$ are compatible if $\nu_A = (\nu_1, \nu_2; \nu_1 + \nu_2)$, $\tau_A = (\tau_1, \tau_2; \nu_1 + \nu_2)$, for some characters ν_i, τ_i of $\mathbf{Z}_p^\times$, and we have the relation*

$$\tau_1 - \tau_2 = \nu_1 - \nu_2 - 2.$$

If ν_A, τ_A are compatible, then taking $\lambda = \nu_1 - \tau_1 = \nu_2 - \tau_2 + 2$, we obtain a homomorphism of $\mathcal{T}_n^\diamond$-representations

$$D^{n-\mathrm{an}}_{G,(\kappa_A+2\rho_{\mathrm{nc}})} \longrightarrow D^{n-\mathrm{an}}_{H,-\tau_A}.$$

Proposition 4.10. *Pairing with $\mathfrak{K}^\lambda$ induces a morphism of sheaves over $\mathcal{U}^H_n$:*

$$(\iota^\sharp)^*(\mathcal{D}^{n-\mathrm{an}}_{G,\nu_A}) \longrightarrow \mathcal{D}^{\mathrm{an}}_{H,\diamond,\tau_A}.$$

This morphism is compatible with specialisation in A, and if $A = \mathbf{Q}_p$ and $\nu = (r_1, r_2; r_1+r_2)$, $\tau = (t_1, t_2; r_1 + r_2)$ are algebraic weights with $r_1 - r_2 \ge 0$ and $r_i, t_i \ge -1$, then this morphism is compatible with the map of finite dimensional sheaves $(\iota^\sharp)^*(\mathcal{V}_{\kappa_2}) \to \mathcal{V}^H_\tau$, where $\mathcal{V}_{\kappa_2}$ is as in [LR23, §2].

In particular, given ν_A and τ_A satisfying $\tau_1 - \tau_2 = \nu_1 - \nu_2 - 2$, we have a morphism of complexes of A-modules

$$(\iota^\sharp)^* : R\Gamma^G_{w,\mathrm{an}}(\nu_A, \mathrm{cusp})^{-,\mathrm{fs}} \longrightarrow R\Gamma_{\mathcal{Z}^H_m}(\mathcal{U}^H_n, \mathcal{D}^{n-\mathrm{an}}_{H,\diamond,\tau_A}(-D_H)). \tag{2}$$

The map ι_p induces in the same way a morphism of sheaves over $\mathcal{U}^H_n$ and an analogous morphism at the level of complexes of A-modules.

4.8. **Locally analytic overconvergent cohomology.** We adopt the same definitions regarding cuspidal, locally analytic, overconvergent cohomology of [LZ21, §9.5]. In particular,

$$R\Gamma^G_{w,\mathrm{an}}(\nu_A, \mathrm{cusp})^{-,\mathrm{fs}} = R\Gamma_{\mathcal{I}^G_{mn}}\left(\mathcal{U}^G_n, \mathcal{D}^{n,-\mathrm{an}}_{G,\nu_A}(-D_G)\right)^{-,\mathrm{fs}},$$

and similarly for the non-cuspidal version. This complex is independent of m, n and t, and is concentrated in degrees $[0,1,2]$.

4.9. **Pairings and duality.** We may define

$$R\Gamma_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{+,\dagger} = \lim_{\to} R\Gamma(\mathcal{Z}^H_{m,\mathrm{Iw}}(p^t), \mathcal{V}^{\mathrm{an}}_{H,\mathrm{Iw},\tau_A}).$$

The following theorem will be crucially used in the definition of the p-adic L-function.

Theorem 4.11. *The cup product induces a pairing*

$$H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A, \mathrm{cusp})^{-,\dagger} \times H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{+,\dagger} \longrightarrow A,$$

whose formation is compatible with base-change in A, and which is compatible with the Serre duality pairing on classical cohomology when $A = \mathbf{Q}_p$ and ν, τ are classical weights.

Proof. The map is defined by combining the pullback maps of (2) for both factors and the pairing between the cohomology groups $H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A, \mathrm{cusp})^{-,\dagger}$ and $H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{+,\dagger}$. By construction, this is compatible with Serre duality for each classical weight. □

4.10. **A Künneth formula for cohomology with support.** In order to define the p-adic L-function, we need to p-adically interpolate the cohomological pairing between H^0 and H^1. This may be regarded as a Künneth formula for cohomology with support.

Proposition 4.12. *The cup product induces a pairing*

$$H^0_{w_0,\mathrm{an}}(\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}(p^t), \tau_1)^{\dagger} \times H^1_{w_1,\mathrm{an}}(\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}(p^t), \tau_2, \mathrm{cusp})^{\dagger} \longrightarrow H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^t), \tau_A)^{-,\dagger},$$

where $\tau_A = (\tau_1, \tau_2)$ is a weight for H.

Proof. This follows as in [LZ21, Thm. 9.6.2] by the general theory of Boxer–Pilloni [BP20, Thm. 6.38]. □

5. The p-adic L-function

In this section we discuss how to use higher Coleman theory to reinterpret the Harris–Su pairing, as discussed in [LR23, §3], in coherent cohomology over certain strata in suitable adic Shimura varieties. In particular, this analysis allows us to perform p-adic interpolation provided that there exist families of cohomology classes interpolating the different elements involved there. We implicitly use some of the results discussed along [LPSZ21, §9,10].

If not specified otherwise, π and σ are cohomological cuspidal automorphic representations of GSp_4 and of GL_2, defined over some field E, both globally generic and unramified outside S. Let L be some p-adic field with an embedding from E.

5.1. **Tame test data.** As in [LZ21, §10.2], we fix the following data:

- M_0, N_0 are positive integers coprime to p with $M_0^2 \mid N_0$, and χ_0 is a Dirichlet character of conductor M_0 (valued in L).
- M_2, N_2 are positive integers coprime to p with $M_2 \mid N_2$, and χ_2 is a Dirichlet character of conductor M_2 (valued in L).

As usual, we use the hat to denote the adelic counterpart of the characters. We consider that the automorphic representations π of G has conductor N_0 and character $\hat{\chi}_0$ (up to twists by norm), and similarly that the representation σ of GL_2 has conductor N_2 and character $\hat{\chi}_2$ (up to twists by norm).

Let S denote the set of primes dividing N_0N_2. By tame test data we mean a pair $\gamma_S = (\gamma_{0,S}, \Phi_S)$, where:

- $\gamma_{0,S} \in G(\mathbf{Q}_S)$, where $\mathbf{Q}_S = \prod_{\ell \in S} \mathbf{Q}_\ell$;
- $\Phi_S \in C_c^\infty(\mathbf{Q}_S^2, L)$, lying in the $(\hat{\chi}_0\hat{\chi}_2)^{-1}$-eigenspace for $\mathbf{Z}_S^\times$.

We let K_S be the quasi-paramodular subgroup of $G(\mathbf{Q}_S)$ of level (N_0, M_0); and we let $\hat{K}_S$ be the open compact subgroup of $G(\mathbf{Q}_S)$ defined in [LZ21, §10.2]. We also use analogous notations for K^p and $\hat{K}^p$, the prime-to-p part of the level and its adelic counterpart, respectively.

5.2. **p-adic families.** As a general piece of notation, we use the conventions regarding p-adic families of [LZ21, §10.4]. In particular, we consider $U \subset \mathcal{W}^2$ an open affinoid disc, and let $\mathbf{r}_1, \mathbf{r}_2 : \mathbf{Z}_p^\times \to \mathcal{O}(U)^\times$ be the universal characters associated to the two factors of $\mathcal{W}^2$. Let ν_U be the character $(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1 + \mathbf{r}_2)$ of $T(\mathbf{Z}_p)$.

Following [LZ21, §10.4], there is a family of automorphic representations $\underline{\pi}$ over the open affinoid disc U introduced at the beginning of the section. Let $S^1(\underline{\pi}) = H^1(\mathcal{M}^{\bullet,-,fs}_{\mathrm{cusp},w_2})$ be the sheaf introduced in Def. 10.4.1 of loc. cit., which is free of rank 1. We shall then choose a basis $\underline{\xi}$ of that space. Since the spaces of higher Coleman theory have an action of $G(\mathbf{A}_{\mathrm{f}}^p)$, we can make sense of $\gamma_{0,S} \cdot \underline{\xi}$ as a family of classes at tame level $\hat{K}^p$, which is still an eigenfamily for the Hecke operators away from S.

Definition 5.1. *We say a point $P \in U(L)$ is nice for $\underline{\pi}$ if the weight of P is $(r_1, r_2) \in U \cap \mathbf{Z}^2$ with $r_1 \geq r_2 \geq 0$ and the specialisation at P of the system of eigenvalues $\lambda^-_{\underline{\pi}}$ attached to the family $\underline{\pi}$ is the character of a p-stabilised automorphic representation π_P, which is cuspidal, globally generic, and has conductor N_0 and character χ_0.*

This implies that the fibre of $S^1(\underline{\pi})$ at P maps isomorphically to the π_P-eigenspace in the classical $H^1(K^p, \kappa_1(\nu), \mathrm{cusp})$; in particular, this eigenspace is 1-dimensional.

Proceeding now as in [LZ21, §10.5], we can consider analogous objects for GL_2. In particular, we may choose a disc $U' \subset \mathcal{W}$ and a finite-slope overconvergent p-adic family of modular eigenforms $\mathcal{G}$ over U' (of weight $\boldsymbol{\ell} + 2$ where $\boldsymbol{\ell}$ is the universal character associated to U'). Then, we say a point $Q \in U'$ is nice for $\mathcal{G}$ if it lies above an integer $\ell \in U' \cap \mathbf{Z}_{\geq 0}$, and the specialisation of $\mathcal{G}$ at Q is a classical form. We further require that the fibre of $S^1(\underline{\sigma})$ at Q maps isomorphically to the σ_Q-eigenspace in the classical H^1 (and in particular, this eigenspace is 1-dimensional). We write σ_ℓ for the corresponding automorphic representation, with the normalisations of loc. cit. As before, we shall take a basis $\underline{\eta}$ of $S^1(\underline{\sigma})$.

Remark 5.2. The inequalities defining region (D) automatically imply that we are not dealing with non-cohomological weights, and hence we do not need to consider an étale covering, as it was the case for region (F). ◇

5.3. **Construction of the imprimitive p-adic L-function.** We refer to [LPSZ21, §7.4] for the construction of p-adic family of Eisenstein series $\mathcal{E}^{\Phi^{(p)}}(0, \mathbf{t} - 1)$, which depends on a prime-to-p Schwartz function $\Phi^{(p)}$. According to [LZ21, Prop. 10.1.2], it is an overconvergent cusp form of weight $\mathbf{t}$.

For our construction, we need to recall the pairing

$$H^0_{w_0,\mathrm{an}}(\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}(p^2), \tau_1)^\dagger \times H^1_{w_1,\mathrm{an}}(\mathcal{S}_{\mathrm{GL}_2,\mathrm{Iw}}(p^2), \tau_2, \mathrm{cusp})^\dagger \longrightarrow H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^2), \tau_A)^{-,\dagger}$$

introduced in Theorem 4.12. From now on, let $A = \mathcal{O}(U \times U')$. Next, we can consider

$$\mathcal{E}^{\Phi^{(p)}}(0, \mathbf{t} - 1) \boxtimes G(\chi_2^{-1})\underline{\eta} \in H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^2), \tau_A)^{+,\dagger},$$

where $\mathbf{t} = \mathbf{r}_2 - \mathbf{r}_1 + \boldsymbol{\ell} - 2$ and the tame level is taken to be $H \cap \hat{K}^p$.

Similarly, let $\underline{\xi}$ be an element in $H^1_{w_{01},\mathrm{an}}(\mathcal{S}_{H,\mathrm{Iw}}(p^2), \tau_A)^{-,\dagger}$.

Definition 5.3. *We let $\mathcal{L}_{p,\gamma_S}(\pi \times \sigma; \underline{\xi}; \underline{\eta})$ denote the element of A defined by*

$$\langle (\iota^\sharp)^*(\gamma_{0,S} \cdot \underline{\xi}), \mathcal{E}^{\Phi^{(p)}}(0, \mathbf{t} - 1) \boxtimes G(\chi_2^{-1})\underline{\eta} \rangle.$$

This is a three-variable p-adic L-function, where we may vary the weights (r_1, r_2) and we keep the linear condition in terms of (r_1, r_2, ℓ, t), namely $\mathbf{t} = \mathbf{r}_2 - \mathbf{r}_1 + \boldsymbol{\ell} - 2$ (alternatively, $s = \frac{\ell - r_1 + r_2 - 2}{2}$).

Definition 5.4.
- *We say a point (P, Q) of $U \times U'$ is nice if $P = (r_1, r_2)$ and $Q = (\ell)$ are integer points, with P nice for $\underline{\pi}$ and Q nice for $\underline{\sigma}$.*
- *We say that (P, Q) is nice critical if we also have $\ell \leq r_1 - r_2 + 1$ (the specialization t at (P, Q) is ≥ 3).*
- *If instead we have $r_1 - r_2 \leq \ell - 2 \leq r_1$, we say that P is nice geometric.*

Remark 5.5. We hope that there should be a p-adic L-function analytically varying in all 4 parameters (the two GSp_4 weights together with the two GL_2 factors), but this cannot be achieved with our methods. ◇

There are two natural questions:

- Along the critical region, can we express the p-adic L-function as an (explicit) multiple of the complex L-value?
- Along the geometric region, can we prove an explicit reciprocity law involving the logarithm of the Euler system of [HJS20]?

In the following sections we give a partial answer to the first one, while the second one will be explored in a forthcoming work.

5.4. **The correction term Z_S.** This section introduces a correction term Z_S which depends on the choice of local data, and which will arise in the interpolation property of the p-adic L-function. Its definition depends on certain Whittaker models properly introduced in [LR23, §6]; since this will have a minor relevance in this work, we just refer the interested reader to our previous paper. In particular, with the notations introduced in loc. cit., we may consider the integral $Z(W, \Phi_1, W^{(\ell)}; s)$.

We shall set

$$Z_S(\pi \times \sigma, \gamma_S; s) = \frac{Z(\gamma_{0,S} \cdot W_0^{\mathrm{new}}, \Phi_S, W_2^{\mathrm{new}}; s)}{G(\chi_2^{-1}) \prod_{\ell \in S} L(\pi_\ell \times \sigma_\ell, s)},$$

and

$$Z_S(\pi \times \sigma, \gamma_S) = Z_S(\pi \times \sigma, \gamma_S; 1 + \frac{t}{2}),$$

where $t = r_2 - r_1 - 2 + \ell$ as usual and $G(\chi_2^{-1})$ is the Gauss sum of the character χ_2. Note that for any given π and σ, one can choose γ_S such that $Z_S(\pi \times \sigma, \gamma_S; s) \neq 0$ (this follows from the definition of the L-factor as a GCD of local zeta-integrals).

5.5. **Choosing the global data.** The global Whittaker transform, given by integrating automorphic forms over the compact quotient $N(\mathbf{Q})\backslash N(\mathbf{A})$, where N is the upper-triangular unipotent subgroup of GSp_4, gives a canonical isomorphism

$$\pi \cong \mathcal{W}(\pi) \cong \otimes_v' \mathcal{W}(\pi_v).$$

For all finite places v, the space $\mathcal{W}(\pi_v)$ has a normalised new-vector w_v^{new}. Hence, given $w_p \in \mathcal{W}(\pi_p)$, we can consider the global Whittaker function

$$w_\infty \cdot w_p \cdot \prod_{v \notin \{p,\infty\}} \gamma_v w_v^{\mathrm{new}},$$

where γ_v is an arbitrary element of $\mathrm{GSp}_4(\mathbf{Q}_v)$ which is the identity if v is unramified, and w_∞ is the standard Whittaker function at ∞. The theory for the σ_i is analogous, with the standard Whittaker function being the complex exponential.

5.6. **Interpolation property.** We choose a $\bar{\mathbf{Q}}$-basis ξ of the new subspace of $H^1(\pi)$, where $H^1(\pi)$ is the copy of π_{f} appearing in the degree 1 coherent cohomology of the Siegel Shimura variety. Analogously, we also choose a $\bar{\mathbf{Q}}$-basis η of the new subspace of $H^1(\sigma)$. Comparing $\xi \otimes \eta$ with the standard Whittaker function defines a period $\Omega_\infty(\pi, \sigma) \in \mathbf{C}^\times$.

Definition 5.6. *Given non-zero $\xi \in S^1(\pi, L)$ and $\eta \in S^1(\sigma, L)$, we define periods $\Omega_p(\pi, \sigma) \in L^\times$ and $\Omega_\infty(\pi, \sigma) \in \mathbf{C}^\times$ as in* [LPSZ21, §6.8]. *(These periods do depend on the choices of ξ and η, but we drop that dependence from the notation).*

In the following result we establish the interpolation property for the p-adic L-function; observe that the algebraicity of the right hand side was the main result of [LR23].

Theorem 5.7. *The p-adic L-function $\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})$ has the following interpolation property: if (P, Q) is nice critical, then*

$$\frac{\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})(P, Q)}{\Omega_p(\pi_P, \sigma_Q)} = Z_S(\pi_P \times \sigma_Q, \gamma_S) \cdot \mathcal{E}^{(D)}(\pi_P \times \sigma_Q) \cdot \frac{G(\chi_2^{-1}) \Lambda(\pi_P \times \sigma_Q, s)}{\Omega_\infty(\pi_P, \sigma_Q)},$$

where $\Lambda(\pi_P \times \sigma_Q, s)$ is the completed (complex) L-function.

Proof. By construction, we have

$$\mathcal{L}^{\mathrm{imp}}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})(P, Q) = G(\chi_2^{-1})\langle \hat{\iota}^*(\gamma_{0,S} \cdot \xi_P), \mathcal{E}^{\Phi^{(p)}}(0, t-1) \boxtimes \eta_Q \rangle.$$

Along the region given by $\ell - t = r_1 - r_2 + 2$, this expands as the product of $G(\chi_2^{-1})\Lambda(\pi_P \times \sigma_Q, \frac{t}{2})$ and a product of normalised local zeta-integrals. The local zeta-integral at p has been evaluated in [LR23, §7] and gives the desired Euler factor. The product of zeta-integrals at the bad primes is by definition $G(\chi_2^{-1})Z_S(...)$. □

Remark 5.8. Taking into account the discussions of [LR23, Rmk. 7.14], it is possible to use this same method to get an improved p-adic L-function where the interpolation property involves a degree seven Euler factor.

Further, following the recent works announced by Boxer and Pilloni it should be possible to extend the previous construction to a p-adic L-function in all four variables. ⋄

6. A conjectural reciprocity law

Along this section we assume that π is both Klingen and Siegel ordinary, and that σ is Borel ordinary. This is done just with the purpose of simplifying notations; similar conjectures can be formulated in the more general strictly-small-slope setting, but one needs to use the theory of (φ, Γ)-modules over the Robba ring (rather than actual subrepresentations of Galois representations).

6.1. **Ordinary filtrations at p.** Associated with the family $\underline{\pi}$ we have a family of Galois representations $V(\underline{\pi})$, which is a rank 4 $\mathcal{O}(U)$-module with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, unramified outside pN_0 and with a prescribed trace for Frob_ℓ^{-1}, when $\ell \nmid pN_0$. The Galois representation $V(\underline{\pi})$ has a decreasing filtration by $\mathcal{O}(U)$-submodules stable under $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Borrowing the notations from [LZ21, §11], we write $\mathcal{F}^i V(\underline{\pi})$ for the codimension i subspace, and similarly for its dual $V(\underline{\pi})^*$. Similarly, there is a 2-step filtration for $V(\underline{\sigma})$.

Definition 6.1. *We set*

$$\mathbb{V}^* = V(\underline{\pi})^* \otimes V(\underline{\sigma})(-1 - \mathbf{r}_1);$$

and we let

$$\mathcal{F}^{(D)} V(\underline{\pi} \times \underline{\sigma})^* = (\mathcal{F}^1 V(\underline{\pi})^* \otimes \mathcal{F}^1 V(\underline{\sigma})^*) + (\mathcal{F}^3 V(\underline{\pi})^* \otimes V(\underline{\sigma})^*)$$

and

$$\mathcal{F}^{(E)} V(\underline{\pi} \times \underline{\sigma})^* = (\mathcal{F}^1 V(\underline{\pi})^* \otimes \mathcal{F}^1 V(\underline{\sigma})^*) + (\mathcal{F}^2 V(\underline{\pi})^* \otimes V(\underline{\sigma})^*).$$

For a nice weight (P, Q) we write $\mathbb{V}^*_{P,Q}$ for the specialization of $\mathbb{V}^*$ at (P, Q), so $\mathbb{V}^*_{P,Q} = V(\pi_P)^* \otimes V(\sigma_Q)^*(-1 - r_1)$ if $P = (r_1, r_2)$.

In particular, $\mathcal{F}^{(E)}$ has rank 5, $\mathcal{F}^{(D)}$ has rank 4, and the quotient $\mathrm{Gr}^{(e/d)}$ is isomorphic to

$$\mathrm{Gr}^{(E/D)} \cong (\mathrm{Gr}^2 V(\underline{\pi})^*) \otimes (\mathrm{Gr}^0 V(\underline{\sigma})^*)(-1 - \mathbf{r}_1).$$

6.2. **p-adic periods and p-adic Eichler–Shimura isomorphisms.** The representations $\mathrm{Gr}^2 V(\underline{\pi})(-1 - \mathbf{r}_2)$ and $\mathrm{Gr}^0 V(\underline{\sigma})(1 - \boldsymbol{\ell})$ are unramified, and hence crystalline as $\mathcal{O}(U)$ (resp. $\mathcal{O}(U')$)-linear representations. Since $\mathbf{D}_{\mathrm{cris}}(\mathbf{Q}_p(1))$ is canonically $\mathbf{Q}_p$, we can therefore define $\mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^{(e/d)} \mathbb{V}^*)$ to be an alias for the rank 1 $\mathcal{O}(U \times U')$-module

$$\mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^2 V(\pi)^*(-1 - \mathbf{r}_2)) \hat{\otimes} \mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^0 V(\underline{\sigma})^*(1 - \boldsymbol{\ell})).$$

We can then define a Perrin-Riou big logarithm for $\mathrm{Gr}^{(e/d)} \mathbb{V}^*$, which is a morphism of $\mathcal{O}(U \times U')$-modules

$$\mathcal{L}^{\mathrm{PR}} : H^1(\mathbf{Q}_p, \mathrm{Gr}^{(e/d)} \mathbb{V}^*) \longrightarrow \mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^{(e/d)} \mathbb{V}^*).$$

For nice geometric weights P, this specialises to the Bloch–Kato logarithm map, up to an Euler factor; and for nice critical weights is specialises to the Bloch–Kato dual exponential.

Let P be a nice weight. There is an *Eichler–Shimura* isomorphism

$$\mathrm{ES}^2_{\pi_P} : S^2(\pi_P, L) \cong \mathrm{Gr}^{r_2+1}_{\mathrm{Hdg}} \mathbf{D}_{\mathrm{cris}}(V(\pi_P)) \cong \mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^1(V(\pi_P))).$$

Similarly, for GL_2 we have an isomorphism

$$\mathrm{ES}^1_{\sigma_Q} : S^1(\sigma_Q, L) \cong \mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^0 V(\sigma_Q)).$$

In this case, the existence of a comparison in families is known after Kings–Loeffler–Zerbes [KLZ17], that is, there exists an isomorphism of $\mathcal{O}(U')$-modules

$$\mathrm{ES}^0_{\underline{\sigma}} : S^0(\underline{\sigma}) \cong \mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^1 V(\underline{\sigma}))$$

interpolating the isomorphism $\mathrm{ES}^1_{\sigma_Q}$ for varying Q, where $\mathcal{S}^1(\underline{\sigma})$ is the $\mathcal{O}(U')$-module spanned by $\underline{\eta}$.

6.3. **Euler system classes.** Suppose that the character $\chi_0\chi_2$ is non-trivial. Then, by the results of [HJS20], associated to the data γ_S, we have a family of cohomology classes

$$\mathbf{z}_m(\underline{\pi} \times \underline{\sigma}, \gamma_S) \in H^1(\mathbf{Q}(\mu_m), \mathbb{V}^*),$$

for all square-free integers coprime to some finite set T containing both p and the ramified primes. The image of $\mathbf{z}_m(\underline{\pi} \times \underline{\sigma}, \gamma_S)$ under localisation at p lands in the image of the injective map from the cohomology of $\mathcal{F}^{(E)}\mathbb{V}^*$ and we can therefore make sense of

$$\mathcal{L}^{\mathrm{PR}}(\mathbf{z}_m(\underline{\pi} \times \underline{\sigma}), \gamma_S) \in \mathbf{D}_{\mathrm{cris}}(\mathrm{Gr}^{(e/f)}\, \mathbb{V}^*).$$

In this setting, we expect the following result.

Conjecture 6.2. *Under the running assumptions, the equality*

$$\left\langle \mathcal{L}^{\mathrm{PR}}(\mathbf{z}_1(\underline{\pi} \times \underline{\sigma}, \gamma_S))(P,Q), \mathrm{ES}^1_{\pi_P}(\xi_P) \otimes \mathrm{ES}^1_{\sigma_Q}(\eta_Q) \right\rangle = \mathcal{L}_{p,\gamma_S}(\underline{\pi} \times \underline{\sigma})(P,Q)$$

holds for all (P,Q) in the geometric range.

The main difficulty for proving the theorem following an analogous strategy to the case of region (F) is the lack of semistable models for the different Shimura varieties involved in this picture (Siegel level). We hope that a better understanding of higher Coleman theory following the new results of Boxer and Pilloni could lead to a proof of the previous conjecture.

References

[Aga07] M. K. Agarwal, *p-adic L-functions for GSp(4) x GL(2)*, ProQuest LLC, Ann Arbor, MI, 2007, Thesis (Ph.D.)–University of Michigan. MR 2711014. ↑ 4

[BP20] G. Boxer and V. Pilloni, *Higher Coleman theory*, preprint, 2020. ↑ 6, 7, 8, 12

[HJS20] C.-Y. Hsu, Z. Jin, and R. Sakamoto, *An Euler system for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, preprint, 2020. ↑ 2, 4, 14, 16

[KLZ17] G. Kings, D. Loeffler, and S. L. Zerbes, *Rankin–Eisenstein classes and explicit reciprocity laws*, Cambridge J. Math. **5** (2017), no. 1, 1–122. MR 3637653. ↑ 16

[LPSZ21] D. Loeffler, V. Pilloni, C. Skinner, and S. L. Zerbes, *Higher Hida theory and p-adic L-functions for* $\mathrm{GSp}(4)$, Duke Math. J. **170** (2021), no. 18, 4033–4121. ↑ 2, 4, 5, 12, 13, 14

[LR23] D. Loeffler and O. Rivero, *Algebraicity of L-values for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, preprint, 2023, `arXiv:2303.16114`. ↑ 1, 2, 3, 4, 8, 11, 12, 14, 15

[LZ20a] D. Loeffler and S. L. Zerbes, *On p-adic regulators for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ *and* $\mathrm{GSp}(4) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$, preprint, 2020. ↑ 4

[LZ20b] ———, *p-adic L-functions and diagonal cycles for* $\mathrm{GSp}(4) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$, in preparation, 2020. ↑ 2, 4

[LZ21] ———, *On the Bloch–Kato conjecture for* $\mathrm{GSp}(4) \times \mathrm{GL}(2)$, preprint, 2021, `arXiv:2106.14511`. ↑ 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15

D.L.: Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom
Email address: `D.A.Loeffler@warwick.ac.uk`

O.R.: Mathematical Sciences Research Institute / Simons Laufer Mathematical Sciences Institute, 17 Gauss Way, Berkeley, CA 94720, USA
Email address: `riverosalgado@gmail.com`