Complex Multiplication

Contents

---

§1. Recall:

E number field, global CFT constructs a map

$$\mathrm{rec}_E : E^\times \backslash \mathbb{A}_E^\times \twoheadrightarrow \mathrm{Gal}(E^{ab}/E)$$

$E^\times \backslash \mathbb{A}_E^\times = C_E$ idele class group

We can be more specific by looking at ray class groups

Given a modulus $\mathfrak{m} = \mathfrak{m}_0 \times \mathfrak{m}_\infty$ of E we can define

$$C_\mathfrak{m} = E^\times \backslash \mathbb{A}_E^\times / U_\mathfrak{m} \qquad \text{ray class group mod } \mathfrak{m}$$

where $U_\mathfrak{m} = \prod_v U_v^{n_v}$

and

$$U_v^{n_v} = \begin{cases} \mathcal{O}_v^\times & v \text{ finite, } n_v = 0 \\ 1 + \varpi_v^{n_v} \mathcal{O}_v & v \text{ finite, } n_v > 0 \\ E_v^\times & v \text{ complex or } v \text{ real, } n_v = 0 \\ \mathbb{R}_{>0}^\times & v \text{ real} \end{cases}$$

Claim: This matches the previously seen definition

$$C_{\mathfrak{m}} = \frac{\{\text{fractional ideals satisfying some condition mod } \mathfrak{m}\}}{\{\text{principal fractional ideals } '' \quad ''\}}$$

CFT: There is a bijection

$$\left\{ \begin{array}{l} \text{Abelian extensions } L/E \text{ of} \\ \text{modulus } \mathfrak{m}_{L/E} \leq \mathfrak{m} \end{array} \right\} \longleftrightarrow \{\text{subgroups of } C_{\mathfrak{m}}\}$$

and isomorphisms

$$\psi_{\mathfrak{m}} : C_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Gal}(E(\mathfrak{m})/E)$$

where $E(\mathfrak{m})$ = ray class field of $\mathfrak{m}$

Important observation:

Let $G = \mathrm{Res}^E_{\mathbb{Q}} \mathbb{G}_m$ a torus

$D = \{id\}$

Then $Sh(G, D) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times D = E^\times \backslash \mathbb{A}^\times_{E,f}$

↑ almost C_E!

and for a level structure K,

$$Sh_K(G, D) = E^\times \backslash \mathbb{A}^\times_{E,f} / K$$

Can choose $K = K_{\mathfrak{m}}$ open, compact s.t.

$$Sh_K(G, D) = E^\times \backslash \mathbb{A}_E^\times / \mathcal{U}_{\mathfrak{m}} = C_{\mathfrak{m}}$$

so C_M is a Shimura variety for the torus!

When $E = \mathbb{Q}$, we can say a lot more:

Theorem: (Kronecker-Weber)

$$\mathbb{Q}^{ab} = \bigcup_{n \geq 1} \mathbb{Q}(\mu_n)$$

This is often called "explicit class field theory"

Kronecker's Jugendtraum:

Can we explicitly characterise E^{ab} for any number field E?

Short answer: Only in special cases

§2. CM Theory for elliptic curves

K imaginary quadratic ext of $\mathbb{Q}$

Recall: There is a 1-1 correspondence

$$\left\{\begin{array}{l}\text{iso classes of} \\ \text{elliptic curves} \\ \text{with torsion data}\end{array}\right\} \longleftrightarrow \Gamma \backslash \mathcal{H}$$

some congruence subgroup Γ

$$\left(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, *\right) \longleftarrow\!\!\!\shortmid \tau$$

$(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, *)$ ⟵ ,

Def: Say $E/\mathbb{C}$ elliptic curve has CM by $\mathcal{O}_K$ if $\text{End}(E) = \mathbb{Z} + f\mathcal{O}_K$ some $f \in \mathbb{N}$

(for convenience write as $\text{End}(E) \simeq \mathcal{O}_K$)

Fact: Given $E/\mathbb{C}$,
either – $\text{End}(E) = \{[m] \mid m \in \mathbb{Z}\}$

or – E has CM by $\mathcal{O}_K$, some imaginary quadratic K

Fix an embedding $K \hookrightarrow \mathbb{C}$, $\Lambda \subseteq \mathcal{O}_K$ ideal.
Then Λ is a lattice in $\mathbb{C}$,
and $E_\Lambda = \mathbb{C}/\Lambda$ is an elliptic curve with
$\text{End}(E_\Lambda) = \{\alpha \in \mathbb{C} \mid \alpha\Lambda = \Lambda\} = \mathcal{O}_K$.

Conversely if $\text{End}(E) \simeq \mathcal{O}_K$, by complex uniformisation
$E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$ for unique $\{\Lambda\} \in Cl(\mathcal{O}_K)$
↑ ideal class group

Prop: There is a 1-1 correspondence
$$Cl(\mathcal{O}_K) \longleftrightarrow \left\{ \begin{array}{l} \text{iso classes of } E/\mathbb{C} \\ \text{with } \text{End}(E) \simeq \mathcal{O}_K \end{array} \right\}$$

Corollary. (a) $\{$iso classes of $E/\mathbb{C}$ with $\text{End}(E) \cong \mathcal{O}_K\}$ is finite

Corollary: (a) $\{$iso classes of $E/\mathbb{C}$ with $\mathrm{End}(E) \cong \mathcal{O}_K\}$ is finite

(b) Let $E/\mathbb{C}$ with $\mathrm{End}(E) \cong \mathcal{O}_K$.

Then $j(E) \in \overline{\mathbb{Q}}$.

Sketch of (b): let $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$,

then $\mathrm{End}(E^\sigma) \cong \mathrm{End}(E) \cong \mathcal{O}_K$

and $j(E^\sigma) = j(E)^\sigma$.

By (a), $\{j(E)^\sigma : \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})\}$ is finite

$\Rightarrow j(E)$ is algebraic.

We can interpret this in terms of Shimura varieties!

Fix an iso $K \cong \mathbb{Q} \oplus \mathbb{Q}$

so $K^\times \longrightarrow GL_2(\mathbb{Q})$

$K^\times = (\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m)(\mathbb{Q})$

Thus we get a map of Shimura varieties

$$K^\times \backslash \mathbb{A}_K^\times \longrightarrow GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H}$$

We can take quotients of appropriate level groups:

e.g. full level of integral points

$$Cl(\mathcal{O}_K) \xrightarrow{\sim} \left\{\begin{matrix}\text{iso classes of elliptic} \\ \text{curves with CM by } \mathcal{O}_K\end{matrix}\right\} \hookrightarrow Y(1)$$

More generally, choose L/K abelian, $P \in E(L)_{tors}$.

$\mathfrak{a}(P)$ is an ideal in $\mathcal{O}_K$.

More generally, choose L/K abelian, ...

Then $\mathfrak{n} := \mathrm{Ann}(P)$ is an ideal in $\mathcal{O}_K$.

Let $N = \min(\mathbb{Z} \cap \mathfrak{n})$

Now we have a map

$$\begin{array}{ccc} C_{\mathfrak{n}} & \longrightarrow & Y_1(N) \\ 1 & \longmapsto & (\mathbb{C}/\mathfrak{n}, 1 \bmod \mathfrak{n}) \end{array}$$

$(\mathbb{C}/\mathfrak{n}, 1 \bmod \mathfrak{n}) =: x$

CM theory tells us $x \in Y_1(N)(\underline{K(\mathfrak{n})})$ is an algebraic point

($K(\mathfrak{n})$: ray class field of $\mathfrak{n}$)

$\therefore$ $C_{\mathfrak{n}} \cong \mathrm{Gal}(K(\mathfrak{n})/K)$

$\therefore$ has a Galois action by multiplication.

$Y_1(N)$ also has a Galois action.

Are these compatible?

Theory of CM tells us yes!

Explicitly, the iso $C_{\mathfrak{n}} \longrightarrow \mathrm{Gal}(K(\mathfrak{n})/K)$ is given by $[\mathfrak{p}] \longmapsto \mathrm{Frob}_{\mathfrak{p}}$ for primes $\mathfrak{p}$

So this gives us an action on $Y_1(N)$ determined by the Artin map!

- Main Theorems of CM for elliptic curves

Using the j-invariant and the above, we can explicitly characterise K^{ab}.

Theorem: (Weber, Fueter) Let $[\Lambda] \in Cl(\mathcal{O}_K)$

(a) $j(\Lambda) \in \overline{\mathbb{Q}}$

(b) $[K(j(\Lambda)) : K] = [\mathbb{Q}(j(\Lambda)) : \mathbb{Q}]$

(c) $H = K(j(\Lambda))$ is the Hilbert Class field of K

(d) If $Cl(\mathcal{O}_K) = \{[\Lambda_1], \dots, [\Lambda_n]\}$, $\{j(\Lambda_1), \dots, j(\Lambda_n)\}$ is a full set of Galois conjugates

We can now use results from global CFT to explicitly write out the Galois action on $j(\Lambda)$:

Theorem (Hasse)**:** Let $[\Lambda] \in Cl(\mathcal{O}_K)$, $H = K(j(\Lambda))$.
Fix $\mathfrak{p} \subseteq \mathcal{O}_K$ prime. If $E/\mathbb{C}$ is any elliptic curve with $j(E) \in H$ and good reduction at primes of H above $\mathfrak{p}$,

$$\mathrm{Frob}_{\mathfrak{p}}(j(\Lambda)) = j(\Lambda \cdot \mathfrak{p}^{-1})$$

- Our next goal: classify K^{ab}

We want to construct K^{ab} by explicitly adding

We want to construct K^{ab} by explicitly adding elements relating to elliptic curves with CM.

Idea: Try torsion points [elliptic curve analogue of roots of unity]

Theorem: Let $E/\mathbb{C}$ be an ell. curve with CM by $\mathcal{O}_K$, $L = K(j(E), E_{tors}) = H(E_{tors})$

Then L/H is abelian

Idea behind proof:

$\mathrm{Gal}(\overline{H}/H)$ and $\mathrm{End}(E) \simeq \mathcal{O}_K$ both give actions on m-torsion for any $m \in \mathbb{Z}$

These actions commute, and $E[m]$ is a free $\mathcal{O}_K/m\mathcal{O}_K$-module of rank 1 $\forall m$

$$\therefore \quad \mathrm{Gal}(H(E[m])/H) \hookrightarrow \mathrm{Aut}(E[m]) \cong (\mathcal{O}_K/m\mathcal{O}_K)^\times$$

which is abelian.

Taking inverse limits

$\Rightarrow \mathrm{Gal}(L/H)$ is abelian

However, this doesn't tell us about K^{ab} when $\# Cl(\mathcal{O}_K) > 1$!

It turns out $H(E_{tors}) \supseteq K^{ab}$, and $\mathrm{Gal}(H(E_{tors})/K^{ab})$ [illegible] of order 2 (usually).

It turns out $H(E_{tors}) \supseteq K^{ab}$, and $\mathrm{Gal}(\ldots / K)$ is a product of groups of order 2 (usually).
So we are not far away!

Solution: fix a map

$$\phi_E : E \longrightarrow E/\mathrm{Aut}(E) \cong \mathbb{P}^1$$

"Weber function"

Theorem: Let $E/\mathbb{C}$ have CM by $\mathcal{O}_K$, then

$$K^{ab} = K(j(E),\ \phi_E(T) : T \in E_{tors})$$

Bonus: We now have a Galois action on E_{tors} using the Artin map! (technically $\phi(E_{tors})$)

§3. CM Theory for abelian varieties

This gives an explicit description of K^{ab} for imaginary quadratic K, but can we say anything for other number fields?

Def: A CM field is a deg 2 extension E/F of a totally real number field F.

Each embedding $F \hookrightarrow \mathbb{R}$ extends to a pair

Each embedding $F \hookrightarrow \mathbb{R}$ extends to a pair
$\varphi, \bar{\varphi} : E \hookrightarrow \mathbb{C}$.

A CM-type Φ for E is a subset
$$\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$$
s.t. $$\Phi \sqcup \bar{\Phi} = \mathrm{Hom}(E, \mathbb{C})$$

Def: Let (E, Φ) be a CM-type. The reflex field $\tilde{E}$ of (E, Φ) is characterised by any of the following equivalent conditions

(a) $\tilde{E}$ = fixed field of $\{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid \Phi^\sigma = \Phi\}$

(b) $\tilde{E} = \mathbb{Q}\left(\sum_{\varphi \in \Phi} \varphi(a) : a \in E\right)$

(c) Smallest subfield of $\bar{\mathbb{Q}}$ s.t. $\exists$ $\tilde{E}$-vector space V and injection $E \hookrightarrow \mathrm{End}_{\tilde{E}}(V)$ for which
$$\mathrm{Tr}(a) = \sum_{\varphi \in \Phi} \varphi(a) \qquad \forall a \in E$$

Remark: If $E/\mathbb{Q}$ is Galois, (b) $\Rightarrow$ $\tilde{E} \subseteq E$.

Def: An abelian variety of CM type (E, Φ) is an abelian variety $A/\mathbb{C}$ of dimension $g = [F:\mathbb{Q}]$ there exists a monomorphism
$$i : E \hookrightarrow \mathrm{End}^\circ(A)$$
and for any $a \in E$,
$$\mathrm{Tr}(a|_{T_0(A)}) = \sum_{\varphi \in \Phi} \varphi(a)$$
($T_0(A)$: tangent space of A at 0)

[In fact: for any such E, i $\exists$ CM-type Φ " above holds.]

[In fact: for any such ... s.t. the above holds.]

Prop: Any abelian variety (A, i) of CM-type (E, Φ) over $\mathbb{C}$ has a model over $\bar{\mathbb{Q}}$.

[For A = elliptic curve this tells us $j(A) \in \bar{\mathbb{Q}}$]

Remarks: (1) From Definition (c) of a reflex field we see that for any field of definition L for A, $L \supseteq \tilde{E}$.
Think of $\tilde{E}$ as the optimal field of definition

(2) The injection $i : E \hookrightarrow \mathrm{End}^0(A)$ is part of the PEL data for the Shimura variety of a unitary group $GU(a,b)$.
Thus we can use these to parametrise abelian varieties with CM.
[Done by Deligne & many others - see Pak-Hin's talk last week & Steven's talk on Siegel modular varieties]

From Milne CM notes

THEOREM 8.17. *Let (G, X) be PEL Shimura datum, as above, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then $\mathrm{Sh}_K(G, X)(\mathbb{C})$ classifies the isomorphism classes of quadruples $((A, i), s, \eta K)$, where*

- A *is a complex abelian variety,*
- $\pm s$ *is a polarization of the Hodge structure $H_1(A, \mathbb{Q})$,*
- i *is a homomorphism $B \to \mathrm{End}^0(A)$, and*
- ηK *is a K-orbit of $B \otimes \mathbb{A}_f$-linear isomorphisms $\eta : V(\mathbb{A}_f) \to V_f(A)$ sending ψ to an $\mathbb{A}_f^{\times}$-multiple of s,*

satisfying the following condition:

$B = K$

Seen in Steven's talk

THEOREM 8.17. *Let (G,X) be PEL Shimura datum, as above, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then $\mathrm{Sh}_K(G,X)(\mathbb{C})$ classifies the isomorphism classes of quadruples $((A,i),s,\eta K)$, where*

- *A is a complex abelian variety,*
- *$\pm s$ is a polarization of the Hodge structure $H_1(A,\mathbb{Q})$,*
- *i is a homomorphism $B \to \mathrm{End}^0(A)$, and*
- *ηK is a K-orbit of $B \otimes \mathbb{A}_f$-linear isomorphisms $\eta\colon V(\mathbb{A}_f) \to V_f(A)$ sending ψ to an $\mathbb{A}_f^\times$-multiple of s,*

satisfying the following condition:

> *(**) there exists a B-linear isomorphism $a\colon H_1(A,\mathbb{Q}) \to V$ sending s to a $\mathbb{Q}^\times$-multiple of ψ, and for such an isomorphism $a \circ h_A \circ a^{-1} \in X$.*

$B = K$

Such in Steven's talk

§4 Main Theorem of CM

Suppose V is an $\bar{E}$-vector space with action of E, s.t. $\mathrm{Tr}_{\bar{E}}(a|_V) = \sum_{\varphi \in \Phi} \varphi(a) \quad \forall a \in E.$

Define the reflex norm by

$$N_\Phi : \mathrm{Res}^E_{\mathbb{Q}} \mathbb{G}_m \longrightarrow \mathrm{Res}^E_{\mathbb{Q}} \mathbb{G}_m$$
$$a \longmapsto \det_E(a|_V)$$

Main Theorem: Let (A,i) be an abelian

Main Theorem: Let (A, i) be an abelian variety of CM-type (E, Φ) over $\mathbb{C}$, and $\sigma \in \mathrm{Aut}(\mathbb{C}/\tilde{E})$.

Given $s \in \mathbb{A}^{\times}_{\tilde{E},f}$ with $rec_{\tilde{E}}(s) = \sigma|_{\tilde{E}^{ab}}$,

$\exists$ unique E-linear isogeny $\alpha : A \to A^{\sigma}$ s.t.

$$\alpha(N_{\Phi}(s) \cdot x) = \sigma x \qquad \forall x \in V_f A$$

Essentially: We can characterise the action of $\mathrm{Gal}(\tilde{E}^{ab}/\tilde{E})$ on torsion of A using reciprocity map!

Moreover if $\Lambda \subseteq E$ a lattice, $u : E \otimes \mathbb{R} \longrightarrow \mathbb{C}^g$ group hom s.t. $A \cong \mathbb{C}^n/u(\Lambda)$, then $A^{\sigma} \cong \mathbb{C}^n/u(N_{\Phi}(s)^{-1}\Lambda)$

Corollary: The isogeny class of (A, i) is defined over $\tilde{E}$

Moduli Space Interpretation of the main theorem:

essentially condition (**) in PEL data

THEOREM 9.19 *Let σ be an automorphism of $\mathbb{C}$ fixing E^*. If $(A, j, \lambda, \eta K)$ satisfies (*), then so also does $\sigma(A, j, \lambda, \eta K)$. Moreover, the isomorphism class of $\sigma(A, j, \lambda, \eta K)$ depends only on $\sigma|E^{*\mathrm{ab}}$. For any $s \in \mathbb{A}^{\times}_{f,E^*}$ such that $\mathrm{art}_{E^*}(s) = \sigma|E^{*\mathrm{ab}}$,*

$$\sigma(A, j, \lambda, \eta K) \approx (A, j, \lambda, \eta f K), \text{ where } f = N_{\Phi}(s).$$

$\mathrm{Gal}(\tilde{E}^{ab}/\tilde{E})$ acts on iso classes of abelian varieties

So $\mathrm{Gal}(\tilde{E}^{ab}/\tilde{E})$ acts on iso classes of abelian varieties in terms of the Artin map too.

Remark: The main theorem of CM gives us no info on abelian extensions of F but gives everything else.

More precisely,

Theorem: (Weil) Verlagerung map

$\mathrm{Ver}_{L/K} = \mathrm{Art}_L \circ \chi_{cyc}$

if $H = \mathrm{im}\left(\mathrm{Gal}(\bar{F}/F) \xrightarrow{\mathrm{Ver}_{E/F}} \mathrm{Gal}(\bar{E}/E)\right)$,

$$E\left(\begin{array}{c}\text{special values of automorphic functions} \\ \text{on canonical models of Shimura varieties} \\ \text{of rational weight}\end{array}\right) = (E^{ab})^H \, \mathbb{Q}^{ab}$$