STARK-HEEGNER POINTS
-PART 2-

Talk 8 of the study group

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§ 1. Previous lectures

$x, y \in \mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, f wt 2 m.f. on $\mathcal{H}_p \times \mathcal{H}$.

Define $\widetilde{\kappa_f}\{x,y\} : \mathcal{E}(\mathcal{T}) \longrightarrow \mathbb{C}$

$$e \longmapsto c \cdot 2\pi i \int_x^y f_e(z)\, dz$$

- We saw it is a harmonic cocycle. Gives rise to complex valued distribution on the boundary $\mathbb{P}^1(\mathbb{Q}_p)$ of $\mathcal{H}_p$, by $\widetilde{\mu_f}\{x \to y\}(U_e) = \widetilde{\kappa_f}\{x \to y\}(e)$, where U_e attached to e via the map Elvira & Muhammad described.
- We also saw it takes values in Λ_E so indeed we have a measure.

Def.: $$\int_{z_1}^{z_2} \int_x^y \omega := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t-z_2}{t-z_1}\right) d\mu_f\{x \to y\}(t),$$

$$\times\!\!\!\!\int_{z_1}^{z_2} \int_x^y \omega := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t-z_2}{t-z_1}\right) d\mu_f\{x \to y\}(t).$$

§2. Some p-adic cocycles

$\tau \in \mathcal{H}_p$.

<u>Aim</u>: attach to τ $\quad \Theta_\tau \in \mathrm{Hom}(\underset{\substack{\text{stab. of } \tau \\ \text{in } \Gamma.}}{\Gamma_\tau}, \mathbb{C}_p^\times / q^{\mathbb{Z}})$

Consider a 2-cochain $\kappa_\tau \in Z^2(\Gamma, \mathbb{C}_p^\times)$ by choosing a base point $x \in \mathbb{P}^1(\mathbb{Q}_p)$ and setting

$$\kappa_\tau(\gamma_1, \gamma_2) = \times\!\!\!\!\!\!\int_\tau^{\gamma_1 \tau} \int_{\gamma_1 x}^{\gamma_1 \gamma_2 x} \omega.$$

<u>Fact</u>. It is indeed a 2-cocycle + image in $H^2(\Gamma, \mathbb{C}_p^\times)$ depends only on f (not on x, nor on τ).

<u>Conj.</u> Let $q \in \mathbb{Q}_p^\times$ be the Tate period attached to E. The image of κ_τ in $H^2(\Gamma, \mathbb{C}_p^\times / q^{\mathbb{Z}})$ is 0.

↳ Relation w/ other well established conjectures of Mazur, Tate and Teitelbaum.

<u>Corollary</u> $\exists\, \tilde{J}_\tau \in C^1(\Gamma, \mathbb{C}_p^\times / q^{\mathbb{Z}})$ st $\kappa_\tau = d J_\tau$ mod $q^{\mathbb{Z}}$.

$\tilde{J}_\tau$ well-defined up to an element of $Z^1(\Gamma, \mathbb{C}_p^\times / q^{\mathbb{Z}})$ $= \mathrm{Hom}(\Gamma, \mathbb{C}_p^\times / q^{\mathbb{Z}})$.

<u>Thm</u> [Ihara] The abelianisation of Γ is finite, so in particular $H^1(\Gamma, \mathbb{C}_p^\times / q^{\mathbb{Z}})$ is finite.

Let e_Γ be the exponent of $H^1(\Gamma, \mathbb{C}_p^\times/q^{\mathbb{Z}})$.
Then, $\xi_\tau = e_\Gamma \widetilde{\xi_\tau}$ is well-defined.

Dependence on x, τ?

$$K_\tau^x - K_\tau^y = d\rho_\tau^{x,y} \in C^1(\Gamma, \mathbb{C}_p^\times/q^{\mathbb{Z}}),$$

where $\rho_\tau^{x,y}(\gamma) = \times\!\!\!\!\int_\tau^{\gamma\tau}\int_{\gamma x}^{\gamma y} \omega$.

The cochain vanishes on the stab. $\Gamma_\tau \subset \Gamma$.
Then $\theta_\tau := \xi_\tau|_{\Gamma_\tau}$ is independent of $x, \widetilde{\xi_\tau}$.

Then $\theta_\tau \in \mathrm{Hom}(\Gamma_\tau, \mathbb{C}_p^\times/q^{\mathbb{Z}})$

§3. Stark–Heegner points

Want $\Gamma_\tau \neq 0$.

- Recall $\tau \in \mathcal{H}_p' \iff \tau \in K \cap \mathcal{H}_p$ w/ p inert or ramified in K.
- Fix K real quadratic. Modified "Heegner hypothesis" (exchange p and ∞).
  - (1) p inert in K
  - (2) All $p | M$ are split in K.

Let $S_{E,K} = \{p, \infty_1, \infty_2\} \cup \{\lambda | M\}$.

For $\tau \in \mathcal{H}_p \cap K$, define $\mathcal{O}_\tau$ as

$$\mathcal{O}_\tau = \left\{ M \in M_0(M)[\tfrac{1}{p}] \;\middle|\; M\binom{\tau}{1} = a\cdot\binom{\tau}{1} \right\}.$$

<u>Fact</u>. $\mathcal{O}_\tau$ is a $\mathbb{Z}[\frac{1}{p}]$-order in K, and $\mathcal{O}_{\tau,1}^\times$ (elts. of det. 1) has rk. 1.

Further $\Gamma_\tau \simeq \mathcal{O}_{\tau,1}^\times / \langle \pm 1 \rangle$.

Fix $\varepsilon \in \mathbb{C}_p^\times$, fund. unit of norm one in the order $\mathcal{O}_\tau$ and let γ_τ the unique generator of Γ_τ st

$\gamma_\tau \binom{\tau}{1} = \varepsilon \binom{\tau}{1}$.

<u>Def.</u>: $J_\tau = \Theta_\tau(\gamma_\tau) \in \mathbb{C}_p^\times / q^{\mathbb{Z}}$.

Let $P_\tau = \Phi_{Tate}(J_\tau) \in E(\mathbb{C}_p)$.

Define $\Phi_N^{J(p)}(\tau) := P_\tau$.

Fix $\tau \in \mathcal{H}_p'$, let H^+ be the narrow ring class field of K attached to that order. H usual ring class field

H^+/H deg. at most 2 (trivial when $\mathcal{O}_\tau^\times$ has elts of norm -1). Write $Gal(H^+/H) = \langle c \rangle$.

<u>Conj</u> Let $\tau \in \mathcal{H}_p$. Then

(a) $P_\tau = \Phi^{J(p)}(\tau) \in E(H^+)$

(b) $cP_\tau = w_\infty P_\tau$.

$\hookrightarrow$ + Shimura rec. law to describe the Galois action.

§ 4. Computations

J_τ explicit? Need the 1-cochain $\tilde{J}_\tau$ st $d\tilde{J}_\tau = \kappa_\tau$.

<u>Theory of mod. symbols</u>

$c_\tau \in C^1(\Gamma, \mathcal{M}_0(\mathbb{C}_p^\times))$.

$$c_\tau(\gamma)(\{x\} - \{y\}) = \times\!\!\!\!\!\!\int_\tau^{\gamma\tau} \int_x^y \omega$$

$$\longrightarrow [c_\tau] \in H^1(\Gamma, \mathcal{M}_0(\mathbb{C}_p^\times))$$

$$\downarrow \delta$$

$$H^2(\Gamma, \mathbb{C}_p^\times)$$

(coming from $0 \to \mathbb{C}_p^\times \to \underbrace{\mathcal{F}(\mathbb{C}_p^\times)}_{\substack{\text{functions} \\ \text{on } \Gamma_x}} \to \mathcal{M}_0(\mathbb{C}_p^\times) \to 0$).

$\delta([c_\tau]) = [\kappa_\tau^\#]$, where $\kappa_\tau^\#(g_0, g_1) := \kappa_\tau(g_1^{-1}, g_0^{-1})$.

<u>Conj</u> The image of c_τ in $H^1(\Gamma, \mathcal{M}_0(\mathbb{C}_p^\times/q^{\mathbb{Z}}))$ is trivial $\Longrightarrow \exists\, \tilde{\eta}_\tau \in \mathcal{M}_0(\mathbb{C}_p^\times/q^{\mathbb{Z}})$ st $\times\!\!\!\!\!\!\int_\tau^{\gamma\tau}\int_x^y \omega = \tilde{\eta}_\tau\{\gamma^{-1}x \to \gamma^{-1}y\} \div \tilde{\eta}_\tau\{x \to y\}$ $(q^{\mathbb{Z}}$

Let $\eta_\tau = e_\Gamma\, \tilde{\eta}_\tau$

Then, define $\times\!\!\!\!\!\!\int^\tau \int_x^y e_\Gamma\, \omega := \eta_\tau\{x \to y\} \in \mathbb{C}_p^\times/q^{\mathbb{Z}}$

<u>Proposition</u> $\tilde{J}_\tau(\gamma) = \times\!\!\!\!\!\!\int^\tau \int_x^{\gamma x} e_\Gamma\, \omega$.

Then, $J_\tau = \tilde{J}_\tau(\gamma_\tau)$ computed via $\times\!\!\!\!\!\!\int^\tau \int_x^y e_\Gamma\, \omega$.

§5. Results

[BD] Let χ be a genus (quadratic unramified) character of K.

$$\begin{tikzcd} & H_\chi = \mathbb{Q}(\sqrt{D_1},\sqrt{D_2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ \mathbb{Q}(\sqrt{D_1}) \arrow[dr, dash, "\chi_1"'] & K \arrow[d, dash, "\epsilon_K"'] & \mathbb{Q}(\sqrt{D_2}) \arrow[dl, dash, "\chi_2"] \\ & \mathbb{Q} & \end{tikzcd}$$

$G = \mathrm{Gal}(H/K)$, $\quad P_\chi = \sum_{g \in G} \chi(g) P_\tau^g$.

Let $E(H_\chi)^\chi$ be the submodule of the Mordell-Weil group $E(H_\chi)$ on which $\mathrm{Gal}(H/K)$ acts via χ.

$$\log_E : E(K_p) \longrightarrow K_p$$
$$P \longmapsto \log_q\left(\Phi_{Tate}^{-1}(P)\right).$$

<u>Thm.</u> χ genus character, E at least two primes of mult. red. and $\chi_1(-M) = -w_M$.

① $\exists\, \underline{P}_\chi \in E(H_\chi)^\chi$, $t \in \mathbb{Q}^\times$ s.t.
$$\log_E(P_\chi) = t \log_E(\underline{P}_\chi)$$

② $\underline{P}_\chi$ has ∞-order iff $L'(E/K, \chi, 1) \neq 0$.

<u>Rk:</u> Then, P_χ coincides w/ the image of a global point in $E(K_p) \otimes \mathbb{Q}$ $\Rightarrow$ integer multiple of $P_\chi \in \mathrm{im}(E(H_\chi)^\chi \to E(K_p))$.

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