Bruhat-Tits tree

19/11/21

1. $\mathbb{H}_p$
2. $\mathcal{T}$: combinatorial definition
3. $PGL_2(\mathbb{Q}_p) \circlearrowright \mathcal{T}$
4. Reduction map: $\mathbb{H}_p \to \mathcal{T}$
5. $\mathcal{E}(\mathcal{T})$ and open sets in $\mathbb{P}^1(\mathbb{Q}_p)$
6. Geometric realisation of $\mathcal{T}$

References: (1-4) Darmon - Rational points on Modular elliptic curves chapter 5

(5-6) Dasgurpta & Teitelbaum: Geometry of the p-adic upper half plane.

1. p-adic upper half plane

fix a prime p

$\mathbb{H}_p := \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ p-adic upper half plane

upper half plane

$\mathbb{H} \rightsquigarrow \mathbb{H}_p$ p-adic analogue

$\circlearrowright$ $PGL_2(\mathbb{Q})$ $\qquad$ $\circlearrowright$ $PGL_2(\mathbb{Q}_p)$ - via Möbius transforms

holomorphic fns $\sim$ opens $\subseteq \mathbb{H}$

rigid analytic fns $\sim$ affinoids $\subseteq \mathbb{H}_p$

$\uparrow$

understand via BT tree

2. Bruhat-Tits tree (combinatorial def^n)

(of $PGL_2(\mathbb{Q}_p)$)

graph: $\mathcal{T} = V(T) \cup \mathcal{E}(T)$
vertices edges

Vertices: homothety classes of $\mathbb{Z}_p$ lattices of $\mathbb{Q}_p^2$

$V(T) \ni [L]$ where $L = \mathbb{Z}_p x_1 + \mathbb{Z}_p x_2 \subseteq \mathbb{Q}_p^2$

with $x_1, x_2 \in \mathbb{Q}_p^2$ linearly independent over $\mathbb{Q}_p$

- $L_1 \sim L_2$: $L_1 = \lambda L_2$ for some $\lambda \in \mathbb{Q}_p^\times$

edges: $\mathcal{E}(T) \ni \{v_1, v_2\}$ adjacent vertices, unordered

v_1, v_2 adjacent: $\exists$ representatives L_1, L_2 of v_1, v_2

such that: $pL_1 \subsetneq L_2 \subsetneq L_1$

* strict inclusions *

Rmk: $pL_1 \subset L_2 \subset L_1 \Rightarrow pL_2 \subset pL_1 \subset L_2$

$\therefore$ undirected graph

Prop: 1) For each $v \in V(\mathcal{T})$: there are $p+1$ adjacent vertices to v

2) $\mathcal{T}$ is connected
3) $\mathcal{T}$ has no loops
$\Big\} \Rightarrow$ "tree" justified

Proof of (1): $v = [L] \in V(T)$

If L' is such that: $pL \subset L' \subset L$

$\Rightarrow L'/pL \subseteq L/pL$

$\therefore L'/pL$ proper subspace of L/pL

L/pL : known - if $L = \mathbb{Z}_p x_1 + \mathbb{Z}_p x_2$:

$\varphi : L \to \mathbb{F}_p^2$, $(y_1 x_1 + y_2 x_2) \mapsto (y_1 \bmod p, y_2 \bmod p)$

$$\left.\begin{array}{l} * \text{ surjective} \\ * \ \ker(\varphi) = pL \end{array}\right\} \Rightarrow L/pL \cong \mathbb{F}_p^2$$

$\therefore$ $L'/pL \cong$ a proper subspace of $\mathbb{F}_p^2$

$\hookrightarrow$ these are the one dimensional subspaces: $\langle a \rangle$

$a \in \{(i,1), (1,0) : i \in \mathbb{F}_p\}$ - $p+1$ pts

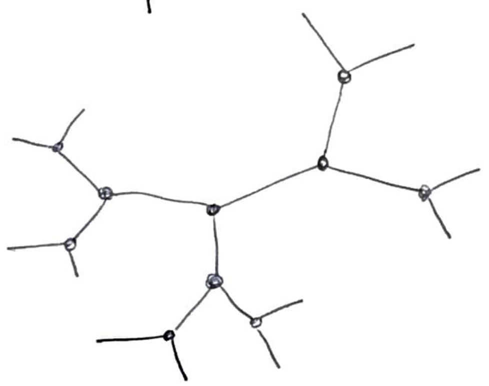
Modulo some checks: Vertices adjacent to v are : $[L_i]$ $i = 0 .. p-1, \infty$

where:

$$L_i = \begin{cases} pL + \mathbb{Z}_p \binom{i}{0} & i = 0 .. p-1 \\ pL + \mathbb{Z}_p \binom{0}{p} & i = \infty \end{cases}$$

$\square$

$p = 2$



$p = 3$

3. $PGL_2(\mathbb{Q}_p)$ action

$$\underset{\substack{\| \\ G}}{PGL_2(\mathbb{Q}_p)} = GL_2(\mathbb{Q}_p)/\mathbb{Q}_p^*$$

define the action: $G \circlearrowright T = V(T) \cup E(T)$

$$\begin{array}{ccc} G \circlearrowright V(T) & : & M \cdot v := [M\mathcal{L}] \\ \end{array}$$

Ψ Ψ

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $v = [\mathcal{L}]$ $\qquad M\mathcal{L} := \{Ml \mid l \in \mathcal{L}\}$

$\mathcal{L} = \mathbb{Z}_p x_1 + \mathbb{Z}_p x_2 \qquad = \mathbb{Z}_p(Mx_1) + \mathbb{Z}_p(Mx_2)$

Well defined: $\mathcal{L}_1 \sim \mathcal{L}_2 \Rightarrow \mathcal{L}_1 = \lambda \mathcal{L}_2$ for some $\lambda \in \mathbb{Q}_p^\times$

$$\Rightarrow M\mathcal{L}_1 = M(\lambda \mathcal{L}_2) = \lambda(M\mathcal{L}_2)$$

also apply M^{-1} for $\Leftarrow$, and thus:

$$L_1 \sim \mathcal{L}_2 \Leftrightarrow M\mathcal{L}_1 \sim M\mathcal{L}_2$$

$G \circlearrowright E(T)$

Ψ Ψ

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $e = \{v_1, v_2\}$ $\qquad M \cdot e := \{Mv_1, Mv_2\}$

Well defined: $\underset{i=1,2}{v_i = [\mathcal{L}_i]}$ adjacent $\Rightarrow p\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_1$

$$\Rightarrow M(p\mathcal{L}_1) = p(M\mathcal{L}_1) \subset M\mathcal{L}_2 \subset M\mathcal{L}_1$$

$\therefore$ v_1, v_2 adjacent $\Leftrightarrow Mv_1, Mv_2$ adjacent

Distinguished vertex : $v_* := [\mathbb{Z}_p^2]$

$$\mathcal{L}^* := \mathbb{Z}_p^2 = \mathbb{Z}_p\begin{pmatrix}1\\0\end{pmatrix} + \mathbb{Z}_p\begin{pmatrix}0\\1\end{pmatrix} \subseteq \mathbb{Q}_p^2$$

adjacent vertices of V_* : $V_i = [\mathcal{L}_i]$ $i = 0 \dots p-1, \infty$

$$\mathcal{L}_i := p\mathcal{L}^* + \mathbb{Z}_p \binom{i}{1} = \mathbb{Z}_p \binom{p}{0} + \mathbb{Z}_p \binom{i}{1} \quad i = 0 \dots p-1$$

$$\mathcal{L}_\infty := p\mathcal{L}^* + \mathbb{Z}_p \binom{1}{0} = \mathbb{Z}_p \binom{1}{0} + \mathbb{Z}_p \binom{0}{p}$$

Distinguished edge: $e_* := \{V_*, V_\infty\}$

Notation: $e_i := \{V_*, V_i\}$ $i = 0 \dots p-1$

Prop: $G \circlearrowright T$ is transitive

Proof: there are 4 parts:

1) $G \circlearrowright V(T)$ transitive
2) $\mathrm{Stab}_G(V_*) = G_0 := PGL_2(\mathbb{Z}_p)$
3) $G_0 \circlearrowright \{e_0 \dots e_{p-1}, e_*\}$ transitive
4) $G \circlearrowright \mathcal{E}(T)$ transitive

1) $\mathcal{L} = \mathbb{Z}_p x_1 + \mathbb{Z}_p x_2 \rightsquigarrow M = (x_1, x_2) \in G$

as x_1, x_2 are linearly independent over $\mathbb{Q}_p$

also : $M\mathcal{L}^* = \mathcal{L}$

take $\underset{\underset{V(T)}{\uparrow}}{v_1, v_2}$ represented by $\mathcal{L}_1, \mathcal{L}_2$

$\Rightarrow \exists M_1, M_2 \in G : M_1\mathcal{L}^* = \mathcal{L}_1, M_2\mathcal{L}^* = \mathcal{L}_2$

$\Rightarrow \mathcal{L}^* = M_1^{-1}\mathcal{L}_1$

$\Rightarrow \underbrace{(M_2 M_1^{-1})}_{:= M \in G} \mathcal{L}_1 = M_2\mathcal{L}^* = \mathcal{L}_2$

2) $\mathrm{Stab}_G(V_*) \supseteq G_0$ clear

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Stab}_G(V_*) \Rightarrow$ WLOG: $M\mathcal{L}^* = \mathcal{L}^*$ multiply by a scalar if necessary

$M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \in L^* = \mathbb{Z}_p^2, \quad M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix} \in L^* = \mathbb{Z}_p^2$

3) 2 cases to consider:

$$\left.\begin{aligned} L_a &= \mathbb{Z}_p\begin{pmatrix}p\\0\end{pmatrix} + \mathbb{Z}_p\begin{pmatrix}a\\1\end{pmatrix} \\ L_b &= \mathbb{Z}_p\begin{pmatrix}p\\0\end{pmatrix} + \mathbb{Z}_p\begin{pmatrix}b\\1\end{pmatrix}\end{aligned}\right\} \quad M := \begin{pmatrix}1 & b-a\\0 & 1\end{pmatrix} \in G_0$$

M fixes V_*

$a, b \in \mathbb{F}_p$

M takes L_a to L_b

$\Rightarrow Me_a = e_b$

$L_a = \mathbb{Z}_p\begin{pmatrix}p\\0\end{pmatrix} + \mathbb{Z}_p\begin{pmatrix}a\\1\end{pmatrix} \quad a \in \mathbb{F}_p$

$L_\infty = \mathbb{Z}_p\begin{pmatrix}0\\p\end{pmatrix} + \mathbb{Z}_p\begin{pmatrix}1\\0\end{pmatrix}$

$\rightsquigarrow M := \begin{pmatrix}a & 1\\1 & 0\end{pmatrix} \in G_0$

M fixes V_*

M takes L_∞ to L_a

$\Rightarrow Me_* = e_a$

4) $$\left.\begin{aligned} e &= \{V_1, V_2\} \\ \tilde{e} &= \{\tilde{V}_1, \tilde{V}_2\}\end{aligned}\right\} \in \mathcal{E}(T)$$

$\Rightarrow \exists M \in G : MV_1 = V_*$

$MV_2 = V_i$ some $i \in \mathbb{F}_p \cup \{*\}$

$\Rightarrow Me = e_i$

$\exists \tilde{M} \in G : \tilde{M}\tilde{V}_1 = V_*$

$\tilde{M}\tilde{V}_2 = V_j$ some $j \in \mathbb{F}_p \cup \{*\}$

$\Rightarrow \tilde{M}\tilde{e} = e_j$

by (3) $\exists M^* \in G_0 : M^* e_i = e_j$

$e \xrightarrow{M} e_i \xrightarrow{M^*} e_j \xrightarrow{\tilde{M}^{-1}} \tilde{e} \quad : \quad A := \tilde{M}^{-1}M^*M \in G$

$Ae = \tilde{e}$ $\square$

Prop: $\mathrm{Stab}_G(e_*) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(\mathbb{Z}_p) \mid c \equiv 0 \bmod p \}$
$= \Gamma_0(p\mathbb{Z}_p)$

Pf: "$\supseteq$" clear

$h_\infty = \mathbb{Z}_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z}_p \begin{pmatrix} 0 \\ p \end{pmatrix} = \{ \begin{pmatrix} a \\ bp \end{pmatrix} : a, b \in \mathbb{Z}_p \}$

$M \in \mathrm{Stab}_G(e_*)$: $M h_* = h_* \Rightarrow M \in PGL_2(\mathbb{Z}_p)$
$M h_\infty = h_\infty$

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in h_\infty \Rightarrow M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \in h_\infty$
$\Rightarrow c \equiv 0 \bmod p$ $\square$

Note: action transitive: stabilisers of other edges are conjugate to $\Gamma_0(p\mathbb{Z}_p)$

4. Reduction $\mathbb{H}_p \to T$

Define: red: $\mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}}_p)$ natural map given by reduction modulo the maximal ideal of the ring of integers of $\mathbb{C}_p$

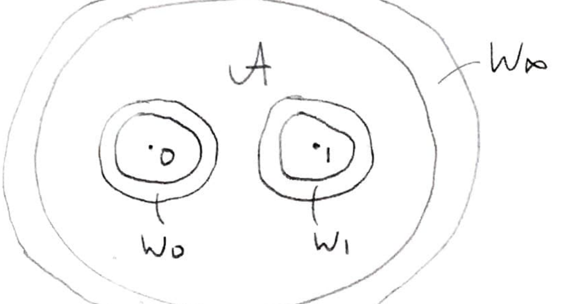
$*$ red$(\mathbb{P}^1(\mathbb{Q}_p)) \subset \mathbb{P}^1(\mathbb{F}_p)$

$\mathcal{A} := \mathrm{red}^{-1}(\mathbb{P}^1(\overline{\mathbb{F}}_p) - \mathbb{P}^1(\mathbb{F}_p)) \subseteq \mathbb{H}_p$ "standard affinoid"
$\|$
$\{ \tau \in \mathbb{H}_p \mid |\tau - t| \geq 1 \text{ for all } t = 0 \ldots p-1, |\tau| \leq 1 \}$

standard annuli:

$W_t = \{ \tau \in \mathbb{H}_p \mid 1/p < |\tau - t| < 1 \}$ $t = 0 \ldots p-1$

$W_\infty = \{ \tau \in \mathbb{H}_p \mid 1 < |\tau| < p \}$



More general affinoids:

$$\begin{array}{ccc} \mathbb{H}_p & & T \\ \circlearrowleft & & \circlearrowleft \\ G = PGL_2(\mathbb{Q}_p) & & G = PGL_2(\mathbb{Q}_p) \\ \text{via Möbius transforms} & & \text{via left translations} \end{array}$$

$\mathcal{A} \subseteq \mathbb{H}_p$

<u>Claim1</u>: $\mathrm{stab}_G \mathcal{A} = G_0 = PGL_2(\mathbb{Z}_p)$

<u>Claim2</u>: $\mathrm{Stab}_G W_\infty = \Gamma_0(p\mathbb{Z}_p)$

<u>Prop</u>: (Reduction map)

There is a unique map: $r: \mathbb{H}_p \to T = V(T) \cup E(T)$

satisfying:

1) $r(\tau) = v_* \Leftrightarrow \tau \in \mathcal{A}$

2) $r(\tau) = e_t \Leftrightarrow \tau \in W_t \quad (W_\infty = W_*)$

$t = 0 \dots p-1, *$

3) r is $PGL_2(\mathbb{Q}_p)$-equivariant: $\boxed{r(\gamma\tau) = \gamma r(\tau)}$

for all $\gamma \in PGL_2(\mathbb{Q}_p)$

[defn + uniqueness follows from (1–3)
Well defined: stabilisers]

$e := \{v_1, v_2\} \in V(T)$

$\mathcal{A}_e := r^{-1}(\{e, v_1, v_2\})$ – standard affinoid attached to e

$W_e := r^{-1}(\{e\})$ – standard annulus attached to e

Note: $\mathcal{A}_e$ is a union of 2 translates by G of $\mathcal{A}$

($\gamma_1^{-1}\mathcal{A}$ and $\gamma_2^{-1}\mathcal{A}$ where $\gamma_1 v_1 = v_0$, $\gamma_2 v_2 = v_0$ $\gamma_i \in G$)

glued along W_e.

* $\mathcal{A}_e$, $e \in \mathcal{E}(T)$: gives a covering of $\mathcal{H}_p$ by standard affinoids where: $\mathcal{A}_{e_1} \cap \mathcal{A}_{e_2} = \emptyset$ or $r^{-1}(v)$ for some $v \in V(T)$

Def: A $\mathbb{C}_p$-valued fn f on $\mathcal{H}_p$ is rigid-analytic if for each edge $e \in \mathcal{E}(T)$: $f|\mathcal{A}_e$ is a uniform limit, with respect to the sup norm, of rational fns on $\mathbb{P}^1(\mathbb{C}_p)$ having poles outside $\mathcal{A}_e$.

5. Edges of T and $\mathbb{P}^1(\mathbb{Q}_p)$

Let: $\Lambda = ([\Lambda_0], [\Lambda_1] \dots)$ be an infinite, non-backtracking sequence of adjacent vertices – a ray of Λ

Two rays: $a = (a_i)$, $b = (b_i)$ are equivalent if they differ by a finite sequence of initial vertices

ie. $a_{n+m} = b_n$ for large enough n, and some fixed m

$E(T) := \{[a] \mid a : \text{ray of } T\}$ – call the ends of T

↑ set of equivalence classes of rays of T

for an oriented edge: $e = ([\Lambda_0], [\Lambda_1])$

fix a source: $[\Lambda_0]$

and a target: $[\Lambda_1]$

$$\mathcal{U}(e) := \{ x \in E(T) : x = ([\Lambda_0], [\Lambda_1] \dots) \}$$

$\mathcal{U}(e)$, $e \in$ oriented edges of T : form a basis of a topology on $E(T)$

Given $x = ([\Lambda_0], [\Lambda_1] \dots) \in E(T)$

Construct a representing sequence of lattices:

$$\Lambda_0 \supsetneq \Lambda_1 \supsetneq \Lambda_2 \supsetneq \dots$$

with: $\Lambda_i / \Lambda_{i+1} \cong \mathbb{Z}_p / p\mathbb{Z}_p$

no backtracking in this sequence: Λ_0 / Λ_i cyclic $\mathbb{Z}_p$-Module of length i, $i \geq 1$

a similar construction: $\Lambda_i / p^i \Lambda_0$: cyclic $\mathbb{Z}_p$-Module of length i, $i \geq 1$

$\Rightarrow \exists\, l_i \in \Lambda_0 \setminus p\Lambda_0 : \Lambda_i = \mathbb{Z}_p l_i + p^i \Lambda_0$

also: $\Lambda_{i+1} = \mathbb{Z}_p l_{i+1} + p^{i+1} \Lambda_0 \subseteq \Lambda_i$

$l_i, l_{i+1} \in \Lambda_0 \setminus p\Lambda_0$

$\Rightarrow l_{i+1} \equiv a\, l_i \mod p^i \Lambda_0$ for some $a \in \mathbb{Z}_p^*$

Thus the l_i form a coherent sequence, converging to an element: $l \in \underbrace{\bigcap_i \Lambda_i}_{\text{one dimensional}}$

Kernel of $l \Rightarrow$ point $N(x) \in \mathbb{P}^1(\mathbb{Q}_p)$

Lemma: $N : \mathrm{End}(X) \to \mathbb{P}^1(\mathbb{Q}_p)$ is a $PGL_2(\mathbb{Q}_p)$-equivariant homeomorphism. $\square$

Pf: Lemma 1.3.6 – Dasguarpta, Teitelbaum

In particular: $N(\mathcal{U}(e)) = \underset{\text{open}}{\mathcal{U} \subseteq} \mathbb{P}^1(\mathbb{Q}_p)$

for any ordered edge e.

by defn of N: $N(\mathcal{U}(e_*)) = \{x \in \mathbb{P}^1(\mathbb{Q}_p) : x \in p\mathbb{Z}_p\}$.

e: any ordered edge

$ge = e_* \Rightarrow N(\mathcal{U}(e)) = \{x \in \mathbb{P}^1(\mathbb{Q}_p) : g^{-1}x \in p\mathbb{Z}_p\}$

$g \in G$

6. Geometric Realisation of $\mathcal{T}$

$X = V(\mathcal{T}) \cup \mathcal{E}(\mathcal{T})$ combinatorially

view each edge of $\mathcal{T}$ as a copy of the unit interval

$\Rightarrow$ we obtain a topological space – called the geometric realisation of $\mathcal{T}$ – denoted by $\mathcal{T}$

a point on an edge of $\mathcal{T}$ joining $[\Lambda]$ and $[\Lambda']$:

$x = (1-t)[\Lambda] + t[\Lambda']$ for some $t \in [0,1]$

– the point "at distance t from $[\Lambda]$ in the direction of $[\Lambda']$"

Rmk: G acts on $\mathcal{T}$, and stabilisers are as before

Def: A norm on $\mathbb{Q}_p^2$ is a function: $\gamma : \mathbb{Q}_p^2 \to \mathbb{R} \cup \{\infty\}$ st:

- $\gamma(x) = \infty \iff x = 0$
- $\gamma(ax) = w(a) + \gamma(x)$, $a \in \mathbb{Q}_p$, $\gamma \in \mathbb{Q}_p^2$

  w: p-adic norm on $\mathbb{Q}_p$
- $\gamma(x+y) \geq \min\{\gamma(x), \gamma(y)\}$

Note: two norms γ_1, γ_2 on $\mathbb{Q}_p^2$ are equivalent if:

$$\underbrace{\gamma_1 - \gamma_2 = c \in \mathbb{R}}_{\text{constant}}$$

Given $x \in T$: we may associate an equivalence class of norms on $\mathbb{Q}_p^2$

Case 1: x is a vertex

$x = [L]$, l_0, l_1 a basis of a representative L

$$\gamma(a l_0 + b l_1) := \min\{w(a), w(b)\}$$

case 2: $x = (1-t)[L] + t[L']$

Choose a basis l_0, l_1 for L s.t: L' is spanned by $l_0, p l_1$.

Define: $\gamma(a l_0 + b l_1) = \min\{w(a), w(b) - t\}$

<u>Prop</u>: This is a bijection between the set of equivalence classes of norms on $\mathbb{Q}_p^2$ and the points of T

Pf: Prop 1.3.4 in Dasgurpta & Teitelbaum $\square$

Rmk: $G \curvearrowright T$ translates to the action on norms by:

$$(g.\gamma)(x) := \gamma(g^{-1}x)$$

This action satisfies the same properties as the action of G on the combinatorial T.

Reduction via Norms:

Given $x \in \mathcal{H}_p$: $x = (a:b) \in \mathbb{P}^1(\mathbb{C}_p)$ - represented by homogeneous coordinates

Define a norm γ_x, up to equivalence, on $\mathbb{Q}_p^2$:

$$\gamma_x(\ell) = \omega(\ell(a,b)) \text{ for a linear form } \ell \text{ on } \mathbb{Q}_p^2$$

- this is a norm (a, b are linearly independent$/\mathbb{Q}_p$)

The map: $x \longmapsto [\gamma_x]$ defines a map $\mathcal{H}_p \to \mathcal{T}$

- called the reduction map (r)

__Lemma__: The reduction map is G-equivariant. Also:

1) $r^{-1}([L^*]) = \{(x:1) \mid x \in \mathbb{C}_p, \omega(x-t) = 0 \ \forall t \in \mathbb{Z}_p\}$

2) $r^{-1}(e_*) = \{(x:1) \mid x \in \mathbb{C}_p, 1 > \omega(x) > 0\}$

Proof: Lemma 1.3.7 - Dasgupta, Teitelbaum $\square$