Last time: Connected Shimura Varieties

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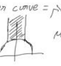
not enough

Recall: Shimura varieties generalize <u>modular curves</u>.

Modular curve $= \Gamma \backslash \mathcal{H}$ $\Gamma \leq SL_2(\mathbb{Z})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\gamma z = \frac{az+b}{cz+d}$



Modular curves are Riemann surfaces

$\therefore$ algebraic curves over $\mathbb{C}$

Non-trivial: they admit natural models over <u>number fields.</u>

E.g.

If $\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$

$\Gamma_0(N) \backslash \mathcal{H}$ an algebraic curve over $\mathbb{Q}$

BUT:

If $\Gamma(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$

$\Gamma(N) \backslash \mathcal{H}$ is <u>not</u> defined over $\mathbb{Q}$

it is defined over $\mathbb{Q}(\zeta_N)$

(actually $\mathbb{Q}(\cos \frac{2\pi}{N})$).

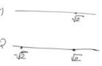
If X/k is a variety, k number field $\to$ we can make it defined over $\mathbb{Q}$ eventually adding connected components.

(Slogan: take the union of the conjugates of X.)

E.g. $X = V(x - \sqrt{2})$ in $\mathbb{A}^1_{\mathbb{Q}(\sqrt{2})}$

$X' = V((x-\sqrt{2})(x+\sqrt{2}))$

$= V(x^2 - 2)$ def'd over $\mathbb{Q}$



<u>Adelic description of modular curves</u>

<u>DEF</u>

The ring of finite Adèles is

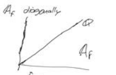
$$\mathbb{A}_f = \prod_{\ell \text{ prime}}' \mathbb{Q}_\ell = \{ (x_\ell) \in \prod \mathbb{Q}_\ell : x_\ell \in \mathbb{Z}_\ell \text{ for } \underbrace{\text{almost all}}_{\text{all but finitely many}} \ell \}$$

We endow $\mathbb{A}_f$ with the topology generated by

$\prod_\ell U_\ell$, $U_\ell \subseteq \mathbb{Q}_\ell$ open

$U_\ell = \mathbb{Z}_\ell$ for almost all ℓ

Remark: $\mathbb{Q}$ embeds in $\mathbb{A}_f$ diagonally



$\mathbb{Q} \subseteq \mathbb{A}_f$ is dense (CRT).

Another way to think of $\mathbb{A}_f$ is:

$$\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_\ell \mathbb{Z}_\ell$$

$\mathbb{A}_f$ = version of $\hat{\mathbb{Z}}$ where we allow finite denom.

$$\boxed{\mathbb{A}_f = \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}}$$

Adèles can be used to give a different description of modular curves.

Observe: If $\Gamma \leq SL_2(\mathbb{Z})$ is congruence (meaning $\Gamma \supseteq \Gamma(N)$ for some N) and $U = \overline{\Gamma}$ in $SL_2(\hat{\mathbb{Z}})$, then U is open and $\Gamma = U \cap SL_2(\mathbb{Z})$.

(<u>This</u> is true more generally: if G is a reductive group and we fix an embedding $G \hookrightarrow GL_n$ then the congruence subgroups of $G(\mathbb{Q})$ are precisely given by $K \cap G(\mathbb{Q})$ where K ranges over compact open subgroups of $G(\mathbb{A}_f)$.)

Let then $\Gamma \leq SL_2(\mathbb{Z})$ be congruence, and $U = \overline{\Gamma}$ in $SL_2(\hat{\mathbb{Z}})$.

$$\Gamma \backslash \mathcal{H} \xrightarrow{\sim} SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}_f) \times \mathcal{H} / U$$

($SL_2(\mathbb{Q})$ acts diagonally; U acts on the first factor)

The map is $\tau \longmapsto (1, \tau)$.

<u>Proof</u>

For surjectivity, let $(g, \tau) \in RHS$

Ug^{-1} is an open nbhd. of g^{-1} in $SL_2(\mathbb{A}_f)$

$\therefore$ since $SL_2(\mathbb{Q}) \subseteq SL_2(\mathbb{A}_f)$ is dense (strong approximation relies on SL_2 SS)

$\exists\, \gamma \in SL_2(\mathbb{Q})$ s.t. $\gamma \in Ug^{-1}$

$\therefore \gamma g \in U$

$(g, \tau) \sim (\gamma g, \gamma \tau) \sim (1, \gamma \tau)$.

Injectivity: Suppose $(1, \tau) \sim (1, \tau')$

$(1, \tau') = (\gamma u, \gamma \tau)$, $\gamma \in SL_2(\mathbb{Q})$, $u \in U$

$\gamma u = 1 \therefore \gamma \in U \cap SL_2(\mathbb{Q}) = \Gamma$

$\tau' = \gamma \tau \sim \tau$ on LHS. ∎

If instead of SL_2 we use GL_2, strong approximation fails ($GL_2(\mathbb{Q}) \subseteq GL_2(\mathbb{A}_f)$ is not dense) so the map not be surjective.

If $Y(U) = GL_2^+(\mathbb{Q}) \backslash GL_2(\mathbb{A}_f) \times \mathcal{H} / U$ then we have the following: $\mathbb{Q}_+^\times \backslash \mathbb{A}_f^\times / \det(U)$ is finite and if $g_1, \ldots, g_n \in GL_2(\mathbb{A}_f)$ s.t. <u>$\det(g_1), \ldots, \det(g_n)$ represent</u> the corresp. classes then

$$\bigsqcup_{i=1}^n \Gamma_{g_i} \backslash \mathcal{H} \xrightarrow{\sim} Y(U)$$

τ is i-th component $\longmapsto (g_i, \tau)$

$\Gamma_{g_i} = GL_2^+(\mathbb{Q}) \cap g_i U g_i^{-1}$

(Note: this is non-canonical)

<u>DEF</u> (Shimura datum)

A <u>Shimura datum</u> is a pair (G, D) where G is a reductive algebraic group$/\mathbb{Q}$ and D is a conjugacy class of homomorphisms $\mathbb{S}_{\mathbb{R}} \to G_{\mathbb{R}}$ s.t.: (D is the set of conjugates of a given h)

(SV1) Only the characters $1, \frac{z}{\bar{z}}, \frac{\bar{z}}{z}$ occur in the representation of $\mathbb{S}_{\mathbb{R}}$ on $\text{Lie}(G^{ad})_{\mathbb{C}}$;

(SV2) $\text{Ad}(h(i))$ is a Cartan involution on $\text{Lie}(G^{ad})_{\mathbb{R}}$;

(SV3) There is no factor H of G^{ad} over $\mathbb{Q}$ s.t. $H(\mathbb{R})$ is compact.

<u>PROP</u>

Let (G, D) is a Shimura datum. Let D_0 be a conn. component of D (prototype: upper half plane). If $K \leq G(\mathbb{A}_f)$ is compact open, then

$$\underbrace{G(\mathbb{Q})_0}\backslash G(\mathbb{A}_f)/K$$

is finite and if C is a set of representatives

$$\bigcup_{c \in C} \Gamma_c \backslash D_0 \xrightarrow{\simeq} \underbrace{G(\mathbb{Q})}\backslash D \times G(\mathbb{A}_f)/K$$

$\Gamma_c = c \cdot K \cdot c^{-1} \cap G(\mathbb{Q})_0$.

Apart from the finiteness statement, the proof is same as before.

<u>DEF</u> (Shimura variety)

Let (G, D) be a Shimura datum. The Shimura variety $Sh(G, D)$ is the inverse system of $Sh_K(G, D) = G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K$ where K ranges over compact subgroups of $G(\mathbb{A}_f)$ s.t. the images of $\Gamma_c = cKc^{-1} \cap G(\mathbb{Q})_0$ in $G^{ad}(\mathbb{Q})$ are torsion-free.

<u>Note</u>

If (G, X) is a Shimura datum and X^+ is a connected component of X. then (G^{der}, X^+) is a connected Shimura datum.

X = conjugacy class of $h: \mathbb{S}_{\mathbb{R}} \to G_{\mathbb{R}}$