TCC COURSE ON IWASAWA THEORY
ASSIGNEMENT 3
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This is the last of the problem sheets for this course, covering material from lectures 5, 6 and 7. Questions are assessed out of a maximum of 100 (although the sum is indeed 110). Students taking this course for credit should submit their solutions to me (by email) by **noon on Friday 24th December**.
This list has three distinct parts, so you can choose which kind of problems are more related with your preferences. The first part (30 points) asks you to go through the results covered during the lectures and fill out some details we skipped. The second part (30 points) asks you to prove different properties of cyclotomic units, some of them also invoked during the lectures, always with a view towards the theory of Euler systems. The third one (25 points) consists on several short questions about p-adic analysis. Finally, The last one (25 points) considers certain twists of the p-adic zeta functions by finite order characters whose conductor is a power of p.
Recall that you only need a minimum of 40 points in each problem sheet.

1 Review of the lectures

Problem 1 (8 points)**.** Let $f \in \Lambda^\times$. Recall the operator $\mathcal{N}$ introduced during the lectures.

(a) Show that $\mathcal{N}(f) \equiv f \pmod{p\Lambda}$.

(b) If $f \equiv 1 \pmod{p^m\Lambda}$, show that $\mathcal{N}(f) \equiv 1 \pmod{p^{m+1}\Lambda}$.

Problem 2 (12 points)**.** The aim of this problem is showing that $\Delta(W) = \Lambda^{\psi=1}$, following the approach suggested in Coates–Sujatha. As a piece of notation, let $\Omega = \Lambda/p\Lambda = \mathbb{F}_p[[T]]$ and let $x \mapsto \tilde{x}$ be the reduction map. If Y is any subset of Λ, we denote by $\tilde{Y}$ its image in Ω under the reduction map.

(a) Show that if $\widetilde{\Delta(W)} = \widetilde{\Lambda^{\psi=1}}$, then $\Delta(W) = \Lambda^{\psi=1}$. *Hint.* Consider any $g \in \Lambda^{\psi=1}$, and let $h_1 \in W$ with $\widetilde{\Delta(h_1)} = \tilde{g}$. Then, $\Delta(h_1) - g = pg_2$. What can you say about the constant term of h_1?

(b) Show that $\widetilde{W} = \Omega^\times$.

Consider now the map

$$\partial \,:\, \Omega^\times \longrightarrow \Omega, \quad \partial(g) = T \cdot \frac{g'(T)}{g(T)}.$$

(c) Show that $\partial(\Omega^\times) = \Phi$, where

$$\Phi = \{f = \sum_{n=1}^{\infty} a_n T^n \text{ such that } a_n = a_{np} \text{ for all } n \geq 1\}.$$

Hint. Justify that $f(T) \in \Omega^\times$ may be written as a convergent infinite product of the form $a\prod_{n=1}^{\infty}(1 - a_nT^n)$, where a is non-zero, and all a_n are in $\mathbb{F}_p$. Further, it may be useful to use the explicit formula

$$\partial(1 - a_kT^k) = -k \sum_{m=1}^{\infty} a_k^m T^{mk}.$$

(d) Show that $T\Omega = \partial(\Omega^\times) + \Theta$, where Θ is the subset of Ω consisting of all series of the form $f = \sum_{n=1}^{\infty} a_nT^n$ with $a_n = 0$ for all n with $(n,p) = 1$.

(e) Show that $\widetilde{\Lambda^{\psi=1}} = \left(\frac{1+T}{T}\right)\partial(\Omega^\times)$.

(f) Conclude that $\Delta(W) = \Lambda^{\psi=1}$.

(g) Explain with your own words the main implications of this result and the role it plays in our study of the Iwasawa main conjecture.

Problem 3 (10 points)**.** Assume that the class number of $F_0 = \mathbb{Q}(\mu_p)^+$ is prime to p.

(a) Show that $E^1_\infty / C^1_\infty = Y_\infty = 0$.

(b) Show that $X_\infty \simeq \Lambda(G)/I(G)\zeta_p$, and conclude that the main conjecture is true.

2 Cyclotomic units

Problem 4 (10 points)**.** Fix a generator ζ_n of $\mu_{p^{n+1}}$, with the property that $\zeta_{n+1}^p = \zeta_n$ for all $n \geq 0$. Let a and b be non-zero integers which are relatively prime to p, and define

$$\mathbf{u} = (u_n), \qquad \text{where } u_n = \frac{\zeta_n^{-a/2} - \zeta_n^{a/2}}{\zeta_n^{-b/2} - \zeta_n^{b/2}}.$$

Show that u_n is a unit in $\mathbb{Q}(\mu_{p^{n+1}})$ such that $N_{n,m}(u_n) = u_m$ for all $n \geq m$, where $N_{n,m} : \mathbb{Q}(\mu_{p^{n+1}}) \longrightarrow \mathbb{Q}(\mu_{p^{m+1}})$ is the norm map.

Problem 5 (10 points)**.** Show that D_n is generated by all the Galois conjugates of $\pm c_n(e, 1)$, where

$$c_n(e, 1) = \frac{\zeta_n^{-e/2} - \zeta_n^{e/2}}{\zeta_n^{-1/2} - \zeta_n^{1/2}}$$

and the integer e is a primitive root modulo p such that $e^{p-1} - 1$ is not a multiple of p^2.

Problem 6 (10 points)**.** The theory of Euler systems requires similar compatibility relations, but now for all integers m, and not just over the p-th cyclotomic tower. Fixing an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}^\times$, let $\zeta_m = \iota^{-1}(e^{2\pi i/m}) \in \mu_m$. For $m > 1$, set $u_m = 1 - \zeta_m \in \mathbb{Q}(\mu_m)^\times$. For all m (including $m = 1$), set

$$v_m = \begin{cases} u_m & \text{if } p \mid m, \\ N_{\mathbb{Q}(\mu_{pm})/\mathbb{Q}(\mu_m)} u_{pm} & \text{if } p \nmid m. \end{cases}$$

(a) Show that $v_1 = p$.

(b) Show that

$$N_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)} v_{m\ell} = \begin{cases} v_m & \text{if } \ell \mid m \text{ or } \ell = p, \\ (1 - \sigma_\ell^{-1}) \cdot v_m & \text{otherwise.} \end{cases}$$

Here, σ_ℓ is the image of Frob_ℓ in $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$.

3 An introduction to p-adic analysis

Problem 7 (15 points)**.** The p-adic logarithm map $\log_p$ is initially defined on $1 + p\mathbb{Z}_p$ via the usual power series

$$\log_p(1 + z) = \sum_{n \geq 1} (-1)^{n+1} \frac{z^n}{n}.$$

(a) Show that $\log_p(1+z)$ converges in $\mathbb{Z}_p$ for $1+z \in 1+p\mathbb{Z}_p$ and show that if $1+z \in 1+p^n\mathbb{Z}_p$, then $\log_p(1 + z) \in p^n\mathbb{Z}_p$. The logarithm can be extended in many ways to $\mathbb{Q}_p^\times$. A standard *branch* to take is called the *Iwasawa logarithm* and is extended by $\log_p(p) = 0$ and $\log_p(\zeta) = 0$ for $\zeta \in \mu_{p-1}$.

(b) The p-adic exponential map $\exp_p$ is defined by

$$\exp_p(z) = \sum_{n\geq 0} \frac{z^n}{n!}.$$

Show that it converges for $z \in p\mathbb{Z}_p$. Which is the minimal value of $r \in \mathbb{R}$ that allows you to define a p-adic exponential map on the elements of $\mathbb{C}_p$ with valuation $> r$?

(c) Show that for $n \in \mathbb{Z}_{\geq 1}$ (or $n \in \mathbb{Z}_{\geq 2}$ for $p = 2$), $\log_p$ and $\exp_p$ give mutually inverse topological isomorphisms $1 + p^n\mathbb{Z}_p \longrightarrow \mathbb{Z}_p$.

(d) For $s \in \mathbb{Z}_p$, let $\chi_s : 1 + q\mathbb{Z}_p \longrightarrow \mathbb{Z}_p^\times$ be given by

$$\chi_s(u) = u^s = \exp_p(s \log_p(u)).$$

Show that the induced character $\chi_s : \mathbb{Z}_p[[T]] \longrightarrow \mathbb{Z}_p$ is given by

$$\chi_s(f(T)) = f((1+q)^s - 1).$$

Problem 8 (10 points)**.** Let $\mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$ denote the ring of functions from $\mathbb{Z}_p$ to itself.

(a) Show that there is a ring homomorphism $\mathbb{Z}_p[[T]] \longrightarrow \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$ given by

$$f(T) \mapsto L_f(s) := \chi_s(f(T)) = f((1+q)^s - 1).$$

A group $\Gamma = \langle \gamma_0 \rangle \cong \mathbb{Z}_p$ is abelian, so the map $\gamma \mapsto \gamma^{-1}$ is an automorphism $\iota : \Gamma \longrightarrow \Gamma$.

(b) Show that ι induces an automorphism $f \mapsto f^\iota$ of Λ. Determine $f^\iota(T)$.

(c) Show that the μ and λ invariants of f and f^ι are equal.

4 Dirichlet characters and L-functions

The last part of this sheet is devoted to discuss a generalization of the p-adic zeta function, when we allow twists by a Dirichlet character χ. Recall that a *Dirichlet character* $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ is a multiplicative function which has some period $n \geq 1$, and $\chi(a) \neq 0$ for $a \in \mathbb{Z}$ if and only if $(a, n) = 1$. The integer n is called the *modulus* of χ. The *conductor* f_χ of a Dirichlet character χ is the smallest integer f dividing its period such that there exists a Dirichlet character ψ of modulus f with $\chi(a) = \psi(a)$ for all $a \in \mathbb{Z}$ with $(a, n) = 1$. A Dirichlet character is said to be *even* (resp. *odd*) if $\chi(-1) = 1$ (resp. $\chi(-1) = -1$).
The *Dirichlet L-series* attached to χ is the complex-values function on $s \in \mathbb{C}$ defined for $\Re(s) > 1$ by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

and then continued to the whole complex plane.
Let χ be a primitive Dirichlet character of conductor p^n, $n \geq 1$. Define the Gauss sum of χ as

$$G(\chi) := \sum_{c \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(c)\zeta_{p^n}^c,$$

where (ζ_{p^n}) is a primitive p^n-th root of unity.

Problem 9 (15 points)**.** Let μ be a measure on $\mathbb{Z}_p$. Then, the measure μ_χ on $\mathbb{Z}_p$ is defined as

$$\int_{\mathbb{Z}_p} f(x) \cdot \mu_\chi = \int_{\mathbb{Z}_p} \chi(x) f(x) \cdot \mu.$$

(a) Show that $G(\chi)G(\chi^{-1}) = \chi(-1)p^n$ and that

$$G(\chi) = \chi(a) \sum_{c \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(c)\zeta_{p^n}^{ac},$$

for any $a \in \mathbb{Z}_p$. What happens when $a \notin \mathbb{Z}_p^\times$?

(b) Let $\mathcal{M}(\mu)$ denote the Mellin transform of μ. Show that

$$\mathcal{M}(\mu_\chi)(T) = \frac{1}{G(\chi^{-1})} \sum_{c \in (Z/p^n\mathbb{Z})^\times} \chi(c)^{-1}\mathcal{M}(\mu)((1+T)\zeta_{p^n}^c - 1).$$

(c) With the notations of the lectures, consider the measure $\mu = \mu_a$ which in particular corresponds to $F_a(T) = \frac{1}{T} - \frac{a}{(1+T)^a - 1}$ under the Mahler transform. Find the Mahler transform of the measure obtained via the above process, denoted as $\mu_{\chi,a}$.

Problem 10 (10 points)**.** Let

$$f_{\chi,a} = \frac{1}{G(\chi^{-1})} \sum_{c \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(c)^{-1}\Big(\frac{1}{e^t\zeta_{p^n}^c - 1} - \frac{a}{e^{at}\zeta_{p^n}^{ac} - 1}\Big).$$

(a) Show that $L(f_{\chi,a}, s) = \chi(-1)(1 - \chi(a)a^{1-s})L(\chi, s)$, where

$$L(f_{\chi,a}, s) = \frac{1}{\Gamma(s)} = \int_0^\infty f_{\chi,a}(t)\, dt^{s-1}\, dt.$$

(b) Let χ be an even character. Show that $L(\chi, -k) = 0$ for all even k. What happens when χ is odd?