Siegel Modular variety

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§ Symplectic spaces & cplx structures

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§ Symplectic spaces & cplx structures

<u>Def</u> A <u>symplectic space</u> is a pair (V, ψ)

- V 2n-dimensional vsp/k
- ψ nondegenerate alternating pairing $V \times V \to k$

<u>Def</u> <u>General symplectic group</u> : symplectic similitudes of (V, ψ)

$$G = GSp(\psi) = \{g \in GL(V) : \psi(gu, gv) = \nu(g)\, \psi(u,v)\}$$

<u>Symplectic group</u> $Sp = Sp(\psi) = \ker(\nu) = \{g \in GSp(\psi) : \nu(g) = 1\}$

$$G^{der} = Sp \qquad G^{ad} = Sp/\{\pm 1\} \qquad Z(G) = \mathbb{G}_m$$

<u>E.g.</u> $\dim V = 2 \Rightarrow$ unique ψ $\Rightarrow$ every transformation preserves it

$$GSp(\psi) = GL_2$$
$$Sp(\psi) = SL_2$$

<u>Def</u> A <u>symplectic complex structure</u> is $J \in Sp(\mathbb{R})$ s.t. $J^2 = -Id$.

This gives $\psi_J : (u,v) \longmapsto \psi(u, Jv)$ symmetric bilinear

Def J is *positive* if ψ_J is pos. def.

$D^+ := \{$positive ~~cplx~~ symplectic cplx str$\}$
$D^- :$ " negative
$D := D^+ \cup D^-$

$G_{\mathbb{R}}$ acts on D: $(g, J) \mapsto gJg^{-1}$

$\mathrm{Stab}_{G_{\mathbb{R}}}(D^+) = G(\mathbb{R})_0 \quad (\nu(g) > 0)$

$G_{\mathbb{R}}$ acts transitively on D
$Sp_{\mathbb{R}}$ " " " D^+

J gives a map $\quad S_{\mathbb{R}} = \mathbb{C}^\times \longrightarrow G_{\mathbb{R}}$
$\quad a+bi \longmapsto a+bJ$

$D \longleftrightarrow \{G_{\mathbb{R}}$-conj classes of maps $\mathbb{C}^\times \to G_{\mathbb{R}}\}$
$J \longmapsto h_J$
$h(i) \longleftarrow h$

§ Siegel Shimura datum & Siegel modular variety

Thm (G, D) is a Shimura datum. Shimura datum attached to (V, ψ)

Pf: $GSp(\psi)$ is reductive LAG / $\mathbb{Q}$
D : $G_{\mathbb{R}}$-conjugacy classes of maps $S_{\mathbb{R}} \to G_{\mathbb{R}}$

SV1: Only the character $1, \frac{z}{\bar z}, \frac{\bar z}{z}$ occur in rep of $S_{\mathbb{R}}$ on $\mathrm{Lie}(G^{ad})_{\mathbb{C}}$
$\mathrm{Lie}(G^{ad})_{\mathbb{C}} \subset \mathrm{End}(V(\mathbb{C}))$ $\quad V = V^+ \oplus V^-$
$h_J(z)$ is mult by z on V^+, by $\bar z$ on V^-.
$\mathrm{End}(V(\mathbb{C})) = \mathrm{End}(V^+) \oplus \mathrm{Hom}(V^+, V^-) \oplus \mathrm{Hom}(V^-, V^+) \oplus \mathrm{End}(V^-)$
$\mathrm{ad}(h_J(z))$ acts as $\quad 1 \quad z/\bar z \quad \bar z/z \quad 1$ ✓

SV2 : $ad(h(i))$ is Cartan involution
$ad(h(i)) : g \mapsto J g J^{-1}$ $\quad J^2 = -Id \Rightarrow$ involution

SV3 : There is no factor H of G^{ad} over $\mathbb{Q}$ s.t. $H(\mathbb{R})$ cpct.

$G^{ad} = Sp/\{\pm 1\}$ is simple over $\mathbb{Q}$
$G^{ad}(\mathbb{R})$ is not compact (~~contains~~ quotient $\mathcal{H}_n$) $\square$

<u>Def</u> <u>Siegel Modular variety attached to (V,ψ)</u> is the Shimura variety of (G,D):

$$Sh(G,D) = \varprojlim_{\substack{K \\ \text{open} \\ \text{cpct}}} \underbrace{G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K}_{Sh_K(G,D)}$$

§ Modular interpretation

<u>Modular curves</u>
- $\Gamma \backslash \mathcal{H}$
- points are elliptic curves.

<u>Def</u> A <u>Hodge structure</u> of type $(-1,0),(0,-1)$ is a pair (W,h)
- W is an $\mathbb{R}$-module
- $h : \mathbb{C}^\times \longrightarrow End(W(\mathbb{R}))$.

$h(i) = J \qquad h_J : a+bi \mapsto a+bJ$.

<u>Def</u> A <u>polarisation</u> on W is a bilinear form $\psi : W \times W \longrightarrow \frac{1}{2\pi i}\mathbb{R}$
- $\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u,v)$
- $\frac{1}{2\pi i}\psi_{\mathbb{R}}(u, Ju) > 0 \quad u \neq 0$

<u>Def</u> Let $K \subset G(\mathbb{A}_f)$ be open cpct. $\mathcal{H}_K$ set of triples $((W,h), s, \eta K)$:
- (W,h) is a rational HS
- s or $-s$ polarisation for (W,h)
- ηK K-orbit of $\mathbb{A}_f$-isoms $V(\mathbb{A}_f) \longrightarrow W(\mathbb{A}_f)$ $\quad s \mapsto \alpha s^{\psi}$

Isom $((W,h), s, \eta K) \xrightarrow{\sim} ((W',h'), s', \eta' K)$ is $b: (W,h) \to (W',h')$
sending s to $\mathbb{Q}^\times$-mult of s', $b \circ \eta = \eta' K$

Prop $\mathcal{H}_K/\approx \longrightarrow G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K = Sh_K(\mathbb{C})$
$((W,h), s, \eta K) \longmapsto [ah, a\circ\eta]$

where $a: W \xrightarrow{\sim} V$ s.t. ψ $\mathbb{Q}^\times$-mult of s.

E.g. $\dim V = 2$. $h \longleftrightarrow$ picking point in upper half plane
$K \longleftrightarrow \Gamma$
$\approx \longleftrightarrow \Gamma$-orbit

Abelian variety $A/\mathbb{C}$ $A = \mathbb{C}^N/\Lambda$ Λ integral HS

Thm $\mathbb{C}/\Lambda$ is ab var $\Leftrightarrow$ $\mathbb{C}/\Lambda$ projective $\Leftrightarrow$ Λ polarisable

Thm (Riemann's theorem) $AV \longrightarrow \{\text{polarisable int HS}\}$
$A \longmapsto \Lambda = H_1(A, \mathbb{Z})$

Cor Let AV° be AV up to isogeny: $\mathrm{Hom}_{AV^\circ}(A,B) = \mathrm{Hom}_{AV}(A,B) \otimes \mathbb{Q}$.

$AV^\circ \xrightarrow{\text{equiv}} \{\text{polarisable Rat}^l \text{ HS}\}$
$A \longmapsto \Lambda \otimes \mathbb{Q} = H_1(A, \mathbb{Q})$

Def $T_f(A) = \varprojlim_N A[N]$ $V_f(A) = T_f(A) \otimes \mathbb{Q}$.
Let M_K be set of triples $(A, s, \eta K)$ s.t.
- A abelian variety $/\mathbb{C}$
- s alternating form on $A \otimes \mathbb{Q}$ s.t. $\pm s$ is a polarisation
- $\eta: V(\mathbb{A}_f) \xrightarrow{\sim} V_f(A)$ s.t. ψ is $\mathbb{A}_f^\times$-mult of s

$(A, s, \eta K) \xrightarrow{\sim} (A', s', \eta' K)$ sends s to $\mathbb{Q}^\times$-mult of s' ηK to $\eta' K$

Thm Canonical bijection $M_K/\approx \longrightarrow Sh_K(\mathbb{C})$

E.g. ηK determine an N-level structure. THE END