Canonical Models of Shimura Varieties : Definitions

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Ref: Milne (12)

Reflex Field "Natural field of defn of a Shimura Variety"

G : reductive group$/\mathbb{Q}$, $k \subseteq \mathbb{C}$ subfield

$\boxed{C(k) := G(k) \backslash \mathrm{Hom}(\mathbb{G}_m, G_k)}$ – conjugacy classes of cocharacters of G_k.

Suppose G splits over k: ie. G_k contains a maximal torus T

Weyl group: $\boxed{W(k) = W(G_k, T) := N(T)/T}$

$N(T)$: normalizer of T in G_k.

Facts: 1. for any field: $k' \supseteq k$: $W(k') = W(k)$

2. the natural map:

$$W(k) \backslash \mathrm{Hom}(\mathbb{G}_m, T_k) \longrightarrow G(k) \backslash \mathrm{Hom}(\mathbb{G}_m, G_k)$$

is a bijection!

Rmk: $\overline{\mathbb{Q}} \subseteq \mathbb{C}$: $W(\overline{\mathbb{Q}}) = W(\mathbb{C})$

$\mathrm{Hom}(\mathbb{G}_m, T_{\overline{\mathbb{Q}}}) = \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})$

$$\therefore \quad \begin{array}{ccc} W(\overline{\mathbb{Q}}) \backslash \mathrm{Hom}(\mathbb{G}_m, T_{\overline{\mathbb{Q}}}) & = & W(\mathbb{C}) \backslash \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}}) \\ \updownarrow 1:1 & & \updownarrow 1:1 \\ C(\overline{\mathbb{Q}}) & \xleftrightarrow{1:1} & C(\mathbb{C}) \end{array} \quad (*)$$

ie. we can identify $C(\overline{\mathbb{Q}})$ and $C(\mathbb{C})$

Let (G, X) be a Shimura datum.

Recall: X = conjugacy class of a hom. $h : (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}} \to G_{\mathbb{R}}$

for any such h, define:

$\mu_h(z) := h(z,1)$ – a cocharacter over $\mathbb{C}$

can be viewed as a composition:

$$\begin{tikzcd} \mu_h : \mathbb{G}_m \arrow[r] & (\mathbb{G}_m \times \mathbb{G}_m) \arrow[r] & (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}} \arrow[d, "h"] \\ z \arrow[r, mapsto] & (z,1) & G_{\mathbb{C}} \end{tikzcd}$$

N.B. a different h will give a $G(\mathbb{C})$-conjugate of μ_h

$\Rightarrow X$ defines an element $c(X) = (\mu_x)_{x \in X} \in C(\mathbb{C})$

$\overset{(*)}{\Rightarrow} c(X) \in C(\overline{\mathbb{Q}})$

Def: the <u>reflex field</u> $E(G,X)$ is the field of defn of $c(X) \in C(\overline{\mathbb{Q}})$

ie. the fixed field of the subgroup of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $c(X)$ (as an element of $C(\overline{\mathbb{Q}})$)

<u>Rmk</u>: 1) $k \subseteq \overline{\mathbb{Q}}$, G splits $/k \Rightarrow E(G,X) \subseteq k$

2) $\mu \in C(k)$ defined$/k \Rightarrow E(G,X) \subseteq k$

3) $i : (G,X) \hookrightarrow (G',X')$ inclusion of Shimura data

$\Rightarrow E(G',X') \subseteq E(G,X)$

EX: Computations of reflex fields

1) $T/\mathbb{Q}$: a torus, $h : S \to T_{\mathbb{R}}$ a hom.

$\mu_h : \mathbb{G}_{m\mathbb{C}} \to T_{\mathbb{C}}$ defined$/\overline{\mathbb{Q}}$

$\Rightarrow E(T,h) =$ field of the subgroup of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing μ_h.

2) (E, Φ) a CM type. $T := (\mathbb{G}_m)_{E/\mathbb{Q}}$.

$T(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \cong (\mathbb{C}^{\Phi})^{\times}$ ie. $E \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^g$

$a \otimes r \mapsto (\varphi_1(a)r \ldots \varphi_g(a)r)$

$\Phi = \{\varphi_1 \ldots \varphi_g\}$

define a homomorphism: $h_\Phi : S \to T_{\mathbb{R}}$

on $\mathbb{R}$ pts: $h_\Phi : \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times$

$z \mapsto (z \cdots z)$ diagonal map

on $\mathbb{C}$ pts: $h_\Phi : S_\mathbb{C} \to T_\mathbb{C} = (\mathbb{C}^\Phi)^\times \times (\mathbb{C}^{\bar\Phi})^\times$

$(z_1, z_2) \mapsto (z_1 \cdots z_1, z_2 \cdots z_2)$

the corresponding cocharacter: $\mu_\Phi : z \mapsto (\underbrace{z \cdots z}_{g \text{ copies}}, \overbrace{1 \cdots 1}^{g \text{ copies}})$

- defined over $\bar{\mathbb{Q}}$
- elements of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ fixing it are those stabilizing Φ

$\Rightarrow E(T, \{h_\Phi\})$ = Reflex field of (E, Φ) – as defined in Muhammad's talk.

<u>Special points</u>

(G, X) a shimura datum

Def: $x \in X$ a <u>special point</u>: if $\exists$ a torus $T \subseteq G$ with:

$\boxed{h_x(\mathbb{C}^\times) \subseteq T(\mathbb{R})}$

We call: (T, x) a <u>special pair</u> in (G, X)

<u>Rmk</u>: • (T, x) special pair $\Rightarrow$ $T(\mathbb{R})$ fixes x ie $\mathrm{ad}(t) \circ h_x = h_x$ $\forall t \in T(\mathbb{R})$

• $\left.\begin{array}{l} T \text{ max. torus in } G \\ T(\mathbb{R}) \text{ fixes } x \end{array}\right\} \Rightarrow h_x(\mathbb{C}^\times) \subseteq T(\mathbb{R})$ ie. (T, x) is a special pair.

<u>EX</u>: $G = GL_2$, $\mathbb{H} = \mathbb{C} \setminus \mathbb{R}$

$G(\mathbb{R}) \curvearrowright \mathbb{H}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \dfrac{az+b}{cz+d}$

pick $z \in \mathbb{C} \setminus \mathbb{R}$ st: $E = \mathbb{Q}(z)$ is quadratic & imaginary.

Consider $E \otimes \mathbb{C}$ as a $\mathbb{C}$-vector space.

$\varphi : E \otimes \mathbb{C} \to \mathbb{C}$

$e \otimes z \mapsto ez$

$\mathrm{Ker}(\varphi) = \langle z \otimes 1 + 1 \otimes (-z) \rangle$

φ is $E \otimes \mathbb{R}$ linear $\Rightarrow$ $(E \otimes \mathbb{R})^\times$ fixes z $\Rightarrow$ z is a special pt.

$$(E \otimes \mathbb{R})^\times = (\mathbb{G}_m)_{E/\mathbb{Q}}(\mathbb{R})$$

max. subtorus of G

. $z \in \mathbb{H}$ special $\Rightarrow$ $\mathbb{Q}(z) \supseteq \mathbb{Q}$ quadratic

Thus: special pts of $\mathbb{H}$ $\longleftrightarrow$ pts $z \in \mathbb{H}$ st: the elliptic curve $\mathbb{C}/\mathbb{Z}+\mathbb{Z}z$ has CM.

Canonical Models

(T, x) special pair of (G, X)

$E(x)$: field of defn of μ_x

We need 2 maps:

from Global Class Field Theory:

$$\mathrm{art}_{E(x)} : \mathbb{A}^\times_{E(x)} \twoheadrightarrow \mathrm{Gal}(E(x)^{ab}/E(x))$$

reciprocal of Artin map / reciprocity map.

define: $r_x : \mathbb{A}^\times_{E(x)} \to T(\mathbb{A}_f)$

$$r_x(a) := \sum_{\rho : E(x) \to \overline{\mathbb{Q}}} \rho(\mu_x(a_f))$$

where: $a = (a_\infty, a_f)$ $\quad a_\infty \in (E(x) \otimes_{\mathbb{Q}} \mathbb{R})^\times$

$a_f \in \mathbb{A}^\times_{E(x),f}$

This is well defined: $\mathbb{A}^\times_{E(x)} \longrightarrow T(\mathbb{A}_{\mathbb{Q}}) \xrightarrow{\text{projection}} T(\mathbb{A}_{\mathbb{Q},f})$

$a_i \mapsto \sum_\rho \rho(\mu_x(a_i)) \in T(\mathbb{A}_{E(x)})$

$\underbrace{\sum_{\rho} \rho(\mu_x(a_i)) \in T(\bar{\mathbb{Q}})}$

stable under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ action $\Rightarrow$ so its an element of $T(\mathbb{Q})$

$K \subseteq G(\mathbb{A}_f)$ compact, open subgroup

$Sh_K(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$

$\quad \Psi$

$[x,a]_K$: with (x,a) as a representative.

Def: a model $M_K(G,X)$ of $Sh_K(G,X)$ over $E(G,X)$ is <u>canonical</u> if for every special pair (T,x) and $a \in G(\mathbb{A}_f)$:

- $[x,a]_K \in M_K(G,X)(\mathbb{C})$ is defined over $E(x)^{ab}$
- $\forall \sigma \in \mathrm{Gal}(E(x)^{ab}/E)$, $s \in \mathbb{A}_{E(x)}^{\times}$ with: $\mathrm{art}_{E(x)}(s) = \sigma$:

$$\boxed{\sigma[x,a]_K = [x, r_x(s)a]_K} \quad (**)$$

Rmk: (T_1,x), (T_2,x) special pairs $\Rightarrow$ $(T_1 \cap T_2, x)$ also a special pair

If $(**)$ holds for one of the 3 $\Rightarrow$ it must hold for all 3.

$\therefore$ in the defn we can restrict to: (T,x) special pair with T a minimal torus with $h_x(\mathbb{S}) \subseteq T_{\mathbb{R}}$

Def: (G,X) Shimura datum

- a <u>model</u> of $Sh(G,X)$ over $k \subseteq \mathbb{C}$ is an inverse system: $M(G,X) = (M_K(G,X))_K$ of varieties $M_K(G,X)$ over k st:

with right $G(\mathbb{A}_f)$ action

$M(G,X)_{\mathbb{C}} = Sh(G,X)$

- a model $M(G,X)$ over $E(G,X)$ is <u>canonical</u> if each $M_K(G,X)$ is canonical.

Thm: For every Shimura datum (G,X), $Sh(G,X)$ has a canonical model, and this model is unique up to canonical isomorphism.

Note: k a field of char. 0.

We have an equivalence of categories:

$$\left\{\begin{matrix}\text{0-dim varieties} \\ \text{over } k\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finite sets with cts} \\ Gal(\bar{k}/k) \text{ action}\end{matrix}\right\}$$

i.e. action factors through $Gal(L/k)$ for some Galois extension $k \subseteq L$, with $L \subset \bar{k}$.

V: 0-dim variety $/\mathbb{C} \longleftrightarrow$ a finite set

$\therefore$ To give a model of V over a number field E: give a continuous action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\underbrace{V(\mathbb{C})}_{\text{a finite set of pts.}}$

EX: Tori
- T a torus$/\mathbb{Q}$
- $h: \mathbb{S} \to T_{\mathbb{R}}$ a homomorphism

$\}$ (T,h) is a Shimura datum.

$E := E(T,h)$ field of defn of μ_h.

$Sh_K(T,h) = T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f)/K$ – is a finite set

- $(**)$ defines a cts action of $Gal(E^{ab}/E)$ on $Sh_K(T,h)$
  - this action defines a model of $Sh_K(T,h)$ over E, which is canonical by definition.

EX: CM Tori

(E,Φ) a CM type $\longrightarrow$ (T, h_Φ) as previously defined

$E(T,h_\Phi) = E^*$ reflex field of the CM type

$$r(T,\mu_\Phi) : (\mathbb{G}_m)_{E^*/\mathbb{Q}} \to (\mathbb{G}_m)_{E/\mathbb{Q}}$$

$$r(T,\mu_\Phi)(P) := \sum_{\rho: E^* \to \bar{\mathbb{Q}}} \rho(\mu_\Phi(P)) \quad = N_{\Phi^*}(P) \quad \text{"reflex norm"}$$

$N_{\Phi^*}(P) = \det(P)$.

$K \subseteq T(\mathbb{A}_f)$ compact and open.

$\mathcal{M}_K$: set of isomorphism classes of $(A, i, \eta K)$ where:

- (A,i): abelian variety / $\mathbb{C}$ of CM type (E, Φ)
- η : $E \otimes \mathbb{A}_f$ linear isomorphism $V(\mathbb{A}_f) \to V_f(A)$

An isomorphism : $(A, i, \eta K) \xrightarrow{\sim} (A', i', \eta' K)$ is an E-linear iso. $A \to A'$, sending ηK to $\eta' K$.

<u>Claim</u>: $Sh_K(T, h_\Phi)$ classifies $\mathcal{M}_K$.

<u>Sketch</u>: Let V be a 1-dim E vector space.

E acts on $V \Rightarrow$ we can view T as a subtorus of $GL(V)$.

- (A,i) of CM type $(E, \Phi) \Rightarrow \exists$ an E-isomorphism
  $a : H_1(A, \mathbb{Q}) \to V$
  - explicitly given in Milne (10).

The isomorphism : $V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{a} V(\mathbb{A}_f)$

is $E \otimes \mathbb{A}_f$-linear $\Rightarrow$ so its multiplication by an element:
$g \in (E \otimes \mathbb{A}_f)^\times = T(\mathbb{A}_f)$

$\therefore$ $(A, i, \eta) \mapsto [g]$ defines a bijection $\mathcal{M}_K \to Sh_K(T, h_\Phi)$

We also have an equivalence of categories:

Abelian varieties / $\bar{\mathbb{Q}}$ of CM type (E, Φ) $\longleftrightarrow$ " " / $\mathbb{C}$ " "

$\therefore$ $Sh_K(T, h_\Phi)$ classifies iso. classes of $(A, i, \eta K)$ where (A,i): abelian variety / $\bar{\mathbb{Q}}$ of CM type (E, Φ).

Q: Is this a model of $Sh_K(T, h_\Phi)$?
If so, is it canonical?

- $\sigma \in Gal(\overline{\mathbb{Q}}/E^*)$: define an action on $\mathcal{M}_K$ as follows:

$$\sigma(A, i, \eta K) := (\sigma A, {}^{\sigma}i, {}^{\sigma}\eta K)$$

$$\sigma\eta := \sigma \circ \eta : V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{\sigma} V_f(\sigma A)$$

  - σ fixes E^* : $(\sigma A, \sigma i)$ is an abelian variety of CM type (E, Φ)

- $Gal(\overline{\mathbb{Q}}/E^*)$ acts on $Sh_K(T, h_\Phi)$ by (**):

$$\sigma[g] = [r_{h_\Phi}(s) g]_K \quad \text{where } art_{E^*}(s) = \sigma|E^*$$

$\therefore$ this defines a model of $Sh_K(T, h_\Phi)$ over E^*.

This is a canonical model:

<u>Prop</u>: The map : $\mathcal{M}_K \to Sh_K(T, h_\Phi)$, $(A, i, \eta K) \mapsto [a \circ \eta]_K$ commutes with the actions of $Gal(\overline{\mathbb{Q}}/E^*)$

Pf: $\sigma \in Gal(\overline{\mathbb{Q}}/E^*)$

Main thm of CM $\Rightarrow$ $\exists$ E-linear isogeny $\alpha : A \to \sigma A$

$$\alpha(N_{\Phi^*}(s) \cdot x) = \sigma x \quad \forall x \in V_f(A)$$

with $s \in \mathbb{A}_{E^*}$, $art_{E^*}(s) = \sigma|E^*$

$$\sigma(A, i, \eta) \longmapsto [a \circ \alpha^{-1} \circ \sigma \circ \eta]_K \qquad \alpha^{-1} \circ \sigma = N_{\Phi^*}(s) = r_{h_\Phi}(s)$$

$$\Rightarrow [a \circ \alpha^{-1} \circ \sigma \circ \eta]_K = [r_{h_\Phi}(s) \cdot (a \circ \eta)] \qquad \square$$