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Review of Linear Algebraic Groups

References: • Malle, Testerman, Linear Algebraic Groups and Groups of finite type, Ch. 1-9
• Rotger, 1.1

Plan: §1. Basic Definitions
§2. The Lie algebra of a linear algebraic group.
§3. Structure of connected reductive groups
§4. Root Systems.

§1. Basics

Fix an alg. closed field k.

Def: A linear alg. group (LAG) is an affine variety equipped with group structure and the multiplication and inversion are morphisms of varieties

Examples

1) $\mathbb{G}_a = (k, +)$
2) $\mathbb{G}_m = (k^*, +)$.
3) GL_n.

Thm: Every LAG can be embedded as a closed subgroup of GL_n.

Examples

1) T_n = upper triang. matrices
2) U_n = " triag matrices w/ 1s in the diagonal
3) $D_n = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$.
4) SL_n.

Thm: Let G be a linear algebraic group. Then:
1) Irred. components of G are the connected components. and they are disjoint
2) G^0 = conn. comp of the identity, then G^0 is closed normal sgp of G of finite index.

Def: A LAG T is called a *torus* if $T \cong \mathbb{G}_m \times \dots \mathbb{G}_m$, i.e $T \cong D_n$ for some n.

Def: A character of a LAG G is a morphism of alg. groups $\chi: G \to \mathbb{G}_m$
We denote the gp of character $X(G) \to X^\circ(G)$

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co – >> $Y(G) \to X_{\circ}(G)$.

Def: A Borel sgp of a LAG G, $B \leq G$, closed, connected, solvable and maximal w.r.t. these properties.

Example

$G = GL_n$, $T_n = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$, $T_n^- = \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}$.

$D_n = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

Thm: 1) All Borel sgps are conjugate.

2) $B_1, B_2 \leq G$ Borel, then $B_1 \cap B_2 \supset$ max. torus. and the maximal tori are conjugate.

Thm: If G is connected, $B \leq G$ a Borel subgroup, then:

$$G = \bigcup_{g \in G} g^{-1} B g.$$

Def: The radical of G, $R(G)$ is the max. closed, connected, solvable, normal subgroup of G.

The unipotent radical is $R_u(G) \leq R(G)$ $\left(R(G)/R_u(G) \cong D_n \right)$

↳ unipotent.

Def: A reductive group G is LAG s.t. $R_u(G) = 1$.

A semisimple group G – – – s.t. $R(G) = 1$.

Note: $R_u(G) \leq R(G) \leq G^{\circ}$ (the connected comp. of the identity).

Prop: $R(G) = \left(\bigcap_{\substack{B \leq G \\ \text{Borel}}} B \right)^{\circ}$

Example.

$G = GL_n$ is reductive

$R(G) \leq T_n \cap T_n^- = D_n$.

$R_u(G) = 1$, GL_n is reductive

$Z(GL_n) = \{ t I_n \} \cong \mathbb{G}_m$. connected, solvable, normal.

$\Rightarrow GL_n$ is not semisimple

§2. The Lie algebra of a linear Algebraic group.

Let A be a k-algebra

Def: A k-linear map $D: A \to A$ is a derivation if $\forall f, g \in A$:

$$D(fg) = f D(g) + D(f) g.$$

$Der_k(A) = \{ D : A \to A \mid D \text{ a derivation} \}$

Then $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \in Der_k(A)$

so it equips $Der_k(A)$ with the structure of a Lie alg.

Consider: $\lambda_x : k[G] \to k[G]$, $x \in G$. (LAG).

$$(\lambda_x f)(g) = f(x^{-1} g).$$

Def: The Lie algebra of a LAG G is

$$Lie(G) := \{ D \in Der_k(k[G]) \mid D\lambda_x = \lambda_x D \quad \forall x \in G \}.$$

Examples

1) $G = G_a$, $k[G] = k[T]$, D a derivation

$(\lambda_x D)(T) = \lambda_x(p(T)) = p(T - x)$.

$D\lambda_x(T) = D(T - x) = D(T) = p(T)$

$\Rightarrow$ $p(T)$ must be constant

$\rightsquigarrow$ $Lie(G_a) \cong k$.

2) $Lie(G_m) \cong k$.

3) $Lie(GL_n) = gl_n$ (Note: $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is the bracket)

4) $Lie(SL_n) = sl_n = \{ \text{matrices } A \text{ s.t } tr(A) = 0 \}$.

Thm: 1) $Lie(G) = Lie(G^0)$

2) $\dim(G) = \dim(G^0) = \dim(Lie(G))$

3) $Lie(G_1 \times G_2) = Lie(G_1) \oplus Lie(G_2)$.

§3. Structure of Reductive Groups.

G is still a linear alg. group.

The Adjoint representation $Ad : G \to GL(Lie(G))$

$x \mapsto Ad_x$.

$Ad_x : Lie(G) \to Lie(G)$.

$y \mapsto [x, y]$.

Consider a maximal torus $T \leq G$, $\dim(T) \geq 1$.

Set $g = Lie(G)$.

$Ad(T) \leq GL(g)$ is commuting semisimple elts and they can be simult. diagonalized.

Take $\chi \in X(T)$ and define $g_\chi := \{ v \in g \mid (Ad\, t) v = \chi(t) v \quad \forall t \in T \}$.

$$g = \bigoplus_{\chi \in X(T)} g_\chi$$

Def: The set of roots $\Phi(G) = \Phi = \{ \chi \in X(T) \mid \chi \neq 0, \ g_\chi \neq 0 \}$.

Def: The Weyl group (w.r.t. T) $W(G) := W = N_G(T) / C_G(T)$

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Example

$G = GL_n$, max. torus $T = D_n$.

$\mathrm{Lie}(GL_n) = \mathfrak{gl}_n$.

$$\Phi(G) = \{ \chi_{ij} : T \longrightarrow \mathbb{G}_m \}$$
$$\mathrm{diag}(d_1, \dots, d_n) \longmapsto d_i d_j^{-1}$$

$W = (\text{monomial matrices}) / T \cong S_n$.

Prop: $\Phi(G)$ is $W(G)$-stable.

Prop: G is connected LAG. $\qquad R_u(G) = \bigcap_{\substack{B \leq G \\ T \subseteq B}} R_u(B)$

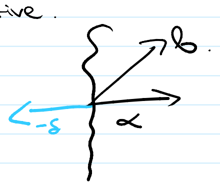
Prop: If G is connected and reductive

1) $\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$, $\dim \mathfrak{g}_\alpha = 1$.

2) $Z(G) = \bigcap_{\alpha \in \Phi(G)} \ker \alpha$.

For G connected + reductive.
Take an elt $\alpha \in \Phi$.

Consider $(s_\alpha) \in W$.



Prop: G connected + reductive (prop. 8.20 in M+T).

$W = \langle s_\alpha \mid \alpha \in \Phi \rangle$.

§4. Root Systems

Def: E a finite dim real v.space and $\Phi \subseteq E$, then E is an abstract root system if

RD1) Φ finite, $0 \notin \Phi$, $\langle \Phi \rangle = E$

RD2) if $c \in \mathbb{R}$, $\alpha, c\alpha \in \Phi$. then $c = \pm 1$

RD3) $\forall \alpha \in \Phi$ $\exists$ reflection $s_\alpha \in GL(E)$, along α, stabilizing Φ

RD4) $\alpha, \beta \in \Phi$, $s_\alpha \beta - \beta$ is an integral multiple of α.

Def: $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ is the Weyl group of Φ.

Def: $\Delta \subseteq \Phi$ base of Φ if it's a vec.space basis of E

Def: $\Delta \subseteq \Phi$ base of Φ if it's a vec. space basis of $\mathcal{E}$ and $\forall \beta \in \Phi$, $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ $c_\alpha \geq 0$ or $c_\alpha \leq 0$.

$\Phi^+ = \{ \beta \mid c_\alpha \geq 0 \}$ system of +ve roots of Φ w.r.t. Δ.

Prop: 1) Φ root system, Δ base exists

2) Δ_1, Δ_2 basis, $w(\Delta_1) = \Delta_2$ (w unique in W).

3) $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$

Def: A root datum $(X, \Phi, Y, \Phi^\vee)$ is a quadruple such that:

RD1) $X \cong \mathbb{Z}^n \cong Y$ and there exists perfect pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$

RD2) $\Phi \subseteq X$ is an abstract root system in $\mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$ and $\Phi^\vee \subseteq Y$ is an abstract root system in $\mathbb{Z}\Phi^\vee \otimes_\mathbb{Z} \mathbb{R}$

RD3) There exists a bijection $\Phi \to \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2 \ \forall \alpha \in \Phi$.

RD4) The reflections s_α of $\alpha \in \Phi$ and $s_{\alpha^\vee}$ of $\alpha^\vee \in \Phi^\vee$ are given by

$$s_\alpha \cdot \chi = \chi - \langle \chi, \alpha^\vee \rangle \alpha, \quad \forall \chi \in X$$

$$s_{\alpha^\vee} \cdot \gamma = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee, \quad \forall \gamma \in Y.$$

Note:
If such a root datum exists, $W(\Phi) = W(\Phi^\vee)$

<u>Example</u>

If G is a connected reductive LAG, then $(X(T), \Phi, Y(T), \Phi^\vee)$ for a maximal torus $T \leq G$ is a root datum.

Def: A root system Φ with base Δ is decomposable if there exists a partition $\Delta = \Delta_1 \sqcup \Delta_2$ into non-empty mutually orthogonal subsets. Otherwise, Φ is indecomposable.

→ Chevalley Classification

Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root data

Moreover, for each root datum, there exists a semisimple algebraic group which realises it. This group is simple if and only if it's root system is indecomposable

An indecomposable root system up to isomorphism is one of the following types:

A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6, E_7, E_8, F_4, G_2